

THE ESSENTIAL NORM OF OPERATORS ON THE BERGMAN SPACE OF VECTOR-VALUED FUNCTIONS ON THE UNIT BALL

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ABSTRACT. Let $A_\alpha^p(\mathbb{B}^n; \mathbb{C}^d)$ be the weighted Bergman space on the unit ball \mathbb{B}^n of \mathbb{C}^n of functions taking values in \mathbb{C}^d . For $1 < p < \infty$ let $\mathcal{T}_{p,\alpha}$ be the algebra generated by finite sums of finite products of Toeplitz operators with bounded matrix-valued symbols (this is called the Toeplitz algebra in the case $d = 1$). We show that every $S \in \mathcal{T}_{p,\alpha}$ can be approximated by localized operators. This will be used to obtain several equivalent expressions for the essential norm of operators in $\mathcal{T}_{p,\alpha}$. We then use this to characterize compact operators in $A_\alpha^p(\mathbb{B}^n; \mathbb{C}^d)$. The main result generalizes previous results and states that an operator in $A_\alpha^p(\mathbb{B}^n; \mathbb{C}^d)$ is compact if only if it is in $\mathcal{T}_{p,\alpha}$ and its Berezin transform vanishes on the boundary.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

1.1. Definition of the Spaces L_α^p and A_α^p . Let \mathbb{B}^n denote the open unit ball in \mathbb{C}^n . Fix some $d \in \mathbb{N}$. If f is a function defined on \mathbb{B}^n taking values in \mathbb{C}^d (that is, f is vector-valued), we say that f is measurable if $z \mapsto \langle f(z), e \rangle_{\mathbb{C}^d}$ is measurable for every $e \in \mathbb{C}^d$. For $\alpha > -1$, let

$$dv_\alpha(z) := c_\alpha(1 - |z|^2)^\alpha dV(z)$$

where dV is volume measure on \mathbb{B}^n and c_α is a constant such that $\int_{\mathbb{B}^n} dv_\alpha(z) = 1$. For vectors in \mathbb{C}^d , let $\|\cdot\|_p$ denote the p -norm on \mathbb{C}^d . That is, if $v = (v_1, \dots, v_d)$ then $\|v\|_p := \left(\sum_{i=1}^d |v_i|^p\right)^{1/p}$. Define $L_\alpha^p(\mathbb{B}^n; \mathbb{C}^d)$ to be the set of all measurable functions on \mathbb{B}^n taking values in \mathbb{C}^d such that

$$\|f\|_{L_\alpha^p(\mathbb{B}^n; \mathbb{C}^d)}^p := \int_{\mathbb{B}^n} \|f(z)\|_p^p dv_\alpha(z) < \infty.$$

It should be noted that $L_\alpha^2(\mathbb{B}^n; \mathbb{C}^d)$ is a Hilbert Space with inner product:

$$\langle f, g \rangle_{L_\alpha^2(\mathbb{B}^n; \mathbb{C}^d)} := \int_{\mathbb{B}^n} \langle f(z), g(z) \rangle_{\mathbb{C}^d} dv_\alpha(z).$$

Similarly, a function f is said to be holomorphic if $z \mapsto \langle f(z), e \rangle_{\mathbb{C}^d}$ is a holomorphic function for every $e \in \mathbb{C}^d$. Since \mathbb{C}^d is a finite dimensional space, this is equivalent to requiring that f be holomorphic in each component function. Define $A_\alpha^2(\mathbb{B}^n; \mathbb{C}^d)$ to be the set of holomorphic functions on \mathbb{B}^n that are also in $L_\alpha^2(\mathbb{B}^n; \mathbb{C}^d)$. Finally, let $\mathcal{L}(L_\alpha^p(\mathbb{B}^n; \mathbb{C}^d))$ denote the bounded linear operators on L_α^p . Define $\mathcal{L}(A_\alpha^p(\mathbb{B}^n; \mathbb{C}^d))$ similarly.

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1.2. Background for the Scalar-Valued Case. For the moment, let $d = 1$. Recall the reproducing kernel:

$$K_z^{(\alpha)}(w) = K^{(\alpha)}(z, w) := \frac{1}{(1 - \bar{z}w)^{n+1+\alpha}}.$$

That is, if $f \in A_\alpha^2(\mathbb{B}^n; \mathbb{C})$ there holds:

$$f(z) = \langle f, K_z^{(\alpha)} \rangle_{A_\alpha^2(\mathbb{B}^n; \mathbb{C})} = \int_{\mathbb{B}^n} \frac{f(w)}{(1 - \bar{z}w)^{n+1+\alpha}} dv_\alpha(w).$$

Recall also the normalized reproducing kernels $k_z^{(p, \alpha)}(w) = \frac{(1-|z|^2)^{\frac{n+1+\alpha}{q}}}{(1-\bar{z}w)^{n+1+\alpha}}$, where q is conjugate exponent to p . There holds that $\|k_z^{(p, \alpha)}\|_{A_\alpha^p(\mathbb{B}^n; \mathbb{C})} \simeq 1$, where the implied constant is independent of z .

The reproducing kernels allow us to explicitly write the orthogonal projection from $L_2^\alpha(\mathbb{B}^n; \mathbb{C})$ to $A_\alpha^2(\mathbb{B}^n; \mathbb{C})$:

$$(P_\alpha f)(z) = \langle f, K_z^{(\alpha)} \rangle_{L_2^\alpha(\mathbb{B}^n; \mathbb{C})}.$$

Let $\phi \in L^\infty(\mathbb{B}^n)$. The Toeplitz operator with symbol ϕ is defined to be:

$$T_\phi := P_\alpha M_\phi.$$

Where M_ϕ is the multiplication operator. So, we have that: $(T_\phi f)(z) = \langle \phi f, K_z^{(\alpha)} \rangle_{L_2^\alpha}$. If T is an operator on $A_\alpha^p(\mathbb{B}^n; \mathbb{C})$, the Berezin transform of T , denoted \tilde{T} is a function on \mathbb{B}^n defined by the formula: $\tilde{T}(\lambda) = \langle T k_\lambda^{(p, \alpha)}, k_\lambda^{(q, \alpha)} \rangle_{A_\alpha^2(\mathbb{B}^n; \mathbb{C})}$.

1.3. Generalization to Vector-Valued Case. Now, we consider $d \in \mathbb{N}$ with $d > 1$. The preceding discussion can be carried over with a few modifications. First, the reproducing kernels remain the same, but the function f is now \mathbb{C}^d -valued and the integrals must be interpreted as vector-valued integrals (that is, integrate in each coordinate). To make this more precise, let $\{e_k\}_{k=1}^d$ be the standard orthonormal basis for \mathbb{C}^d . If f is a \mathbb{C}^d -valued function on \mathbb{B}^n , its integral is defined as:

$$\int_{\mathbb{B}^n} f(z) dv_\alpha(z) := \sum_{k=1}^d \left(\int_{\mathbb{B}^n} \langle f(z), e_k \rangle_{\mathbb{C}^d} dv_\alpha(z) \right) e_k.$$

Let $L_{M^d}^\infty$ denote the set of $d \times d$ matrix-valued functions, φ , such that the function $z \mapsto \|\varphi(z)\|_{\mathbb{C}^d \rightarrow \mathbb{C}^d}$ is in $L^\infty(\mathbb{B}^n; \mathbb{C})$. Note that it is not particularly important which matrix norm is used, since \mathbb{C}^d is finite dimensional and all norms are equivalent. The second change is that the symbols of Toeplitz operators are now matrix-valued functions in $L_{M^d}^\infty$.

Define $\mathcal{T}_{p, \alpha}$ to be the operator-norm topology closure of the set of finite sums of finite products of Toeplitz operators with $L_{M^d}^\infty$ symbols.

Finally, we change the way that we define the Berezin transform of an operator. The Berezin transform will be a matrix-valued function, given by the following relation (see also [1, 13]):

$$\langle \tilde{T}(z)e, h \rangle_{\mathbb{C}^d} = \langle T(k_z^{(p, \alpha)}e), k_z^{(q, \alpha)}h \rangle_{A_\alpha^2} \quad (1)$$

for $e, h \in \mathbb{C}^d$. (Again, q is conjugate exponent to p).

We are now ready to state the main theorem of the paper.

Theorem 1.1. *Let $1 < p < \infty$ and $\alpha > -1$ and $S \in \mathcal{L}(A_\alpha^p, A_\alpha^p)$. Then S is compact if and only if $S \in \mathcal{T}_{p,\alpha}$ and $\lim_{|z| \rightarrow 1} \tilde{S}(z) = 0$.*

1.4. Discussion of the Theorem. By now, there are many results that relate the compactness of an operator to its Berezin transform. It seems that the first result in this direction is due to Axler and Zheng. In [2] they prove that if $T \in \mathcal{L}(A_0^2(\mathbb{B}; \mathbb{C}))$ can be written as a finite sum of finite products of Toeplitz operators, then T is compact if and only if its Berezin transform vanishes on the boundary of \mathbb{B} (recall that $(A_0^2(\mathbb{B}; \mathbb{C}))$ is the standard Bergman space on the unit ball in \mathbb{C}). There are several results generalizing this to larger classes of operators, more general domains, and weighted Bergman spaces. See, for example [7, 9, 17, 22].

There are also several results along these lines for more general operators than those that can be written as finite sums of finite products of Toeplitz operators. In [10] Engliš proves that any compact operator is in the operator-norm topology closure of the set of finite sums of finite products of Toeplitz operators (this is called the Toeplitz algebra). In [26], Suárez proves that an operator, $T \in \mathcal{L}(A_0^p(\mathbb{B}^n; \mathbb{C}))$ is compact if and only if it is in the Toeplitz algebra and its Berezin transform vanishes on $\partial\mathbb{B}^n$. This was extended to the weighted Bergman spaces $A_\alpha^p(\mathbb{B}^n; \mathbb{C})$ in [18] by Suárez, Mitkovski, and Wick and to Bergman spaces on the polydisc and bounded symmetric domains by Mitkovski and Wick in [19] and [20].

2. PRELIMINARIES

We first fix notation that will last for the rest of the paper. The vectors $\{e_i\}_{i=1}^d$, etc. will denote the standard orthonormal basis vectors in \mathbb{C}^d . The letter e will always denote a unit vector in \mathbb{C}^d . For vectors in \mathbb{C}^d , $\|\cdot\|_p$ will denote the l^p norm on \mathbb{C}^d . If M is a $d \times d$ matrix, $\|M\|$ will denote any convenient matrix norm. Since all norms of matrices are equivalent in finite dimensions, the exact norm used does not matter for our considerations. Additionally, $M_{(i,j)}$ will denote the (i, j) entry of M and $E_{(i,j)}$ will be the matrix whose (i, j) entry is 1 and all other entries are 0. Finally, to lighten notation, fix an integer $d > 1$, an integer $n \geq 1$ and a real $\alpha > -1$. Because of this, we will usually suppress these constants in our notation.

2.1. Well-Known Results and Extensions to the Present Case. We will discuss several well-known results about the standard Bergman Spaces, $A_\alpha^p(\mathbb{B}^n; \mathbb{C})$ and state and prove their generalizations to the present vector-valued Bergman Spaces, A_α^p .

Recall the automorphisms, ϕ_z , of the ball that interchange z and 0. The automorphisms are used to define the following metrics:

$$\rho(z, w) := |\phi_z(w)| \quad \text{and} \quad \beta(z, w) := \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

These metrics are invariant under the maps ϕ_z . Define $D(z, r)$ to be the ball in the β metric centered at z with radius r . Recall the following identity:

$$1 - |\phi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.$$

The following change of variables formula is [29, Prop 1.13]:

$$\int_{\mathbb{B}^n} f(w) dv_\alpha(w) = \int_{\mathbb{B}^n} (f \circ \phi_z)(w) |k_z^{(2,\alpha)}(w)|^2 dv_\alpha(w). \quad (2)$$

The following propositions appear in [29].

Proposition 2.1. *If $a \in \mathbb{B}^n$ and $z \in D(a, r)$, there exists a constant depending only on r such that $1 - |a|^2 \simeq 1 - |z|^2 \simeq |1 - \langle a, z \rangle|$.*

Proposition 2.2. *Suppose $r > 0$, $p > 0$, and $\alpha > -1$. Then there exists a constant $C > 0$ such that*

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w)$$

for all holomorphic $f : \mathbb{B}^n \rightarrow \mathbb{C}$ and all $z \in \mathbb{B}^n$.

The following vector-valued analogue will be used:

Proposition 2.3. *Let $\lambda \in \mathbb{B}^n$. There exists a constant $C > 0$ such that*

$$\sup_{z \in D(\lambda, r)} \|f(z)\|_p^p \leq \frac{C}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(\lambda, 2r)} \|f(w)\|_p^p dv_\alpha(w).$$

Proof. First note that $\sup_{z \in D(\lambda, r)} \|f(z)\|_p^p = \sup_{z \in D(\lambda, r)} \sup_{\|e\|_q=1} |\langle e, f(z) \rangle|^p$. By definition, $\langle e, f(z) \rangle_{\mathbb{C}^d}$ is holomorphic for all $e \in \mathbb{C}^d$. By Proposition 2.2 and Proposition 2.1, for $\|e\|_{\mathbb{C}^d} = 1$ and $z \in D(\lambda, r)$ there holds:

$$\begin{aligned} |\langle e, f(z) \rangle_{\mathbb{C}^d}|^p &\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |\langle e, f(w) \rangle_{\mathbb{C}^d}|^p dv_\alpha(w) \\ &\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} \|f(w)\|_p^p dv_\alpha(w) \\ &\simeq \frac{C}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(z,r)} \|f(w)\|_p^p dv_\alpha(w) \\ &\leq \frac{C}{(1 - |\lambda|^2)^{n+1+\alpha}} \int_{D(\lambda, 2r)} \|f(w)\|_p^p dv_\alpha(w). \end{aligned}$$

Which completes the proof. □

The next lemma is in [29]:

Lemma 2.4. *For $z \in \mathbb{B}^n$, s real and $t > -1$, let*

$$F_{s,t}(z) := \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^t}{|1 - \bar{w}z|^s} dv(w).$$

Then $F_{s,t}$ is bounded if $s < n+1+t$ and grows as $(1 - |z|^2)^{n+1+t-s}$ when $|z| \rightarrow 1$ if $s > n+1+t$.

We now give several geometric decompositions of the ball. See [29] for the proofs.

Lemma 2.5. *Given $\varrho > 0$, there is a family of Borel sets $D_m \subset \mathbb{B}^n$ and points $\{w_m\}_{m=1}^\infty$ such that*

- (i): $D(w_m, \frac{\varrho}{4}) \subset D_m \subset D(w_m, \varrho)$ for all m ;
- (ii): $D_k \cap D_l = \emptyset$ if $k \neq l$;

(iii): $\bigcup_{m=1}^{\infty} D_m = \mathbb{B}^n$.

Proposition 2.6. *There exists a positive integer N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}_{k=1}^{\infty}$ in \mathbb{B}^n with the following properties:*

- (i): $\mathbb{B}^n = \bigcup_{k=1}^{\infty} D(a_k, r)$
- (ii): The sets $D(a_k, \frac{r}{4})$ are mutually disjoint.
- (iii): Each point $z \in B_n$ belongs to at most N of the sets $D(a_k, 4r)$.

The following lemma appears in [26].

Lemma 2.7. *Let $\sigma > 0$ and k be a non-negative integer. For each $0 \leq i \leq k$ the family of sets $\mathcal{F}_i = \{F_{i,j} : j \geq 1\}$ forms a covering of \mathbb{B}^n such that*

- (i): $F_{0,j_1} \cap F_{0,j_2} = \emptyset$ if $j_1 \neq j_2$;
- (ii): $F_{0,j} \subset F_{1,j} \subset \dots \subset F_{k,j}$ for all j ;
- (iii): $\beta(F_{i,j}, F_{i+1,j}^c) \geq \sigma$ for all $0 \leq i \leq k$ and $j \geq 1$;
- (iv): every point of \mathbb{B}^n belongs to no more than N elements of \mathcal{F}_i ;
- (v): $\text{diam}_{\beta} F_{i,j} \leq C(k, \sigma)$ for all i, j .

2.2. Matrix-Valued Measures and Their L^p Spaces. We will be concerned with matrix-valued measures, μ . Loosely speaking, a matrix-valued measure is a matrix-valued function on a σ -algebra such that every entry of the matrix is a complex measure. More precisely, a matrix-valued measure is a matrix valued-function, μ , on a σ -algebra such that $\mu(\emptyset) = 0$ and that satisfies countable additivity.

The matrix-valued analogue of non-negative measures are measures such that $\mu(E)$ is a positive semi-definite (PSD) matrix for every Borel subset of \mathbb{B}^n . For every matrix-valued measure, μ , we associate to the matrix its trace measure $\tau_{\mu} := \sum_{i=1}^d \mu_{(i,i)}$. Since the trace of a matrix is the sum of its eigenvalues, and since a PSD matrix has no negative eigenvalues, τ_{μ} is a non-negative scalar-valued measure when μ is a PSD matrix-valued measure. Also, if the trace of a PSD matrix is zero, the matrix is the zero matrix. This implies that $\mu_{(i,j)} \ll \tau_{\mu}$ and so the Lebesgue-Radon-Nikodym derivative, $\frac{d\mu_{(i,j)}}{d\tau_{\mu}}$ is well defined τ_{μ} -a.e.. Let $M_{\mu}(z)$ denote the matrix whose (i, j) entry is $\frac{d\mu_{(i,j)}(z)}{d\tau_{\mu}}$. The following decomposition of the PSD matrix-valued measure μ holds τ_{μ} -a.e.:

$$d\mu(z) = M_{\mu}(z) d\tau_{\mu}(z).$$

If A is a PSD matrix, and $p \geq 1$, we can define a p^{th} -power of A by the following: We have that $A = U^* \Lambda U$ where U is unitary and Λ is diagonal with the eigenvalues of A on the diagonal. Then we define $A^p = U^* \Lambda^p U$. Using this definition, every PSD matrix A has a unique PSD p^{th} -root B given by the formula: $B = U^* \Lambda^{1/p} U$. Consider the following preliminary definition:

Definition 1. Let $L_*^p(\mathbb{B}^n, \mathbb{C}^d; \mu)$ be the set of all \mathbb{C}^d -valued functions that satisfy:

$$\|f\|_{L_*^p(\mathbb{B}^n, \mathbb{C}^d; \mu)}^p := \int_{\mathbb{B}^n} \|M_{\mu}^{1/p}(w) f(w)\|_p^p d\tau_{\mu}(w) < \infty.$$

That $\|f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)}$ is a seminorm is an easy consequence of the fact that $\|\cdot\|_p$ is a norm. However, it is not a norm because if $f(z) \in \ker M(z)$ τ_{μ} -a.e. then $\|f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)} = 0$. We therefore define the following equivalence relation: $f \sim_M g$ if and only if $M(z)f(z) =$

$M(z)g(z)$ τ_μ -a.e. And we define $L^p(\mathbb{B}^n, \mathbb{C}^d; \mu) = L_*^p(\mathbb{B}^n; \mathbb{C}^d, \mu) / \sim_M$. We similarly define $A^p(\mathbb{B}^n; \mathbb{C}^d, \mu)$ to be the set of holomorphic functions that are also in $L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)$.

In the special case $p = 2$, $L^2(\mathbb{B}^n, \mathbb{C}^d; \mu)$ is a Hilbert Space with inner product:

$$\begin{aligned} \langle f, g \rangle_{L^2(\mathbb{B}^n; \mathbb{C}^d, \mu)} &= \int_{\mathbb{B}^n} \langle M_\mu(z)f(z), g(z) \rangle_{\mathbb{C}^d} d\tau_\mu(z) \\ &= \int_{\mathbb{B}^n} \langle d\mu(z)f(z), g(z) \rangle_{\mathbb{C}^d} \end{aligned}$$

There is also the expected Hölder inequality:

Proposition 2.8. *Let μ be a PSD matrix measure on \mathbb{B}^n , $1 < p < \infty$ and q conjugate exponent to p . Then:*

$$\left| \langle f, g \rangle_{L^2(\mathbb{B}^n, \mathbb{C}^d; \mu)} \right| \leq \|f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)} \|g\|_{L^q(\mathbb{B}^n, \mathbb{C}^d; \mu)}.$$

Proof. The proof is a simple computation that uses linear algebra and the usual Hölder's inequality. Indeed,

$$\begin{aligned} \left| \langle f, g \rangle_{L^2(\mathbb{B}^n, \mathbb{C}^d, \mu)} \right| &= \left| \int_{\mathbb{B}^n} \langle M_\mu(w)f(w), g(w) \rangle_{\mathbb{C}^d} d\tau_\mu(w) \right| \\ &\leq \int_{\mathbb{B}^n} |\langle M_\mu^{1/p}(w)f(w), M_\mu^{1/q}(w)g(w) \rangle_{\mathbb{C}^d}| d\tau_\mu(w) \\ &\leq \int_{\mathbb{B}^n} \|M_\mu^{1/p}(w)f(w)\|_{\mathbb{C}^d} \|M_\mu^{1/q}(w)g(w)\|_{\mathbb{C}^d} d\tau_\mu(w) \\ &\leq \left(\int_{\mathbb{B}^n} \|M_\mu^{1/p}(w)f(w)\|_{\mathbb{C}^d}^p d\tau_\mu(w) \right)^{1/p} \left(\int_{\mathbb{B}^n} \|M_\mu^{1/q}(w)g(w)\|_{\mathbb{C}^d}^q d\tau_\mu(w) \right)^{1/q} \\ &= \|f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)} \|g\|_{L^q(\mathbb{B}^n, \mathbb{C}^d; \mu)}. \end{aligned}$$

□

2.3. Matrix-Valued Carleson Measures. We will need to have a concept of matrix-valued Carleson measures. A PSD matrix-valued measure μ on \mathbb{B}^n is a *Carleson matrix-valued measure* for A_α^p if there is a constant C_p , independent of f , such that

$$\left(\int_{\mathbb{B}^n} \|M_\mu^{1/p}(z)f(z)\|_p^p d\tau_\mu(z) \right)^{1/p} \leq C_p \left(\int_{\mathbb{B}^n} \|f(z)\|_p^p dv_\alpha(z) \right)^{1/p}. \quad (3)$$

The best constant for which (3) holds will be denoted by $\|\iota_{(p,d)}\|$. In the case that $p = 2$, the preceding inequality can be written in the following manner:

$$\begin{aligned} \left(\int_{\mathbb{B}^n} \langle d\mu(z)f(z), f(z) \rangle_{\mathbb{C}^d} \right)^{1/2} &\leq C_2 \left(\int_{\mathbb{B}^n} \langle dv_\alpha(z)f(z), f(z) \rangle_{\mathbb{C}^d} \right)^{1/2} \\ &= C_2 \left(\int_{\mathbb{B}^n} \langle f(z), f(z) \rangle_{\mathbb{C}^d} dv_\alpha(z) \right)^{1/2}. \end{aligned}$$

We now to state and give a proof of a Carleson Embedding Theorem for matrix-valued measures. We start by defining a generalization of Toeplitz operators. For μ a matrix-valued measure, define:

$$T_\mu f(z) := \int_{\mathbb{B}^n} \frac{d\mu(w)f(w)}{(1 - \bar{w}z)^{n+1+\alpha}}.$$

Lemma 2.9 (Carleson Embedding Theorem). *For a PSD matrix-valued measure, μ , the following quantities are equivalent:*

- (i): $\|\mu\|_{RKM} := \sup_{e \in \mathbb{C}^d, \|e\|_2=1} \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \langle d\mu(z)e, e \rangle_{\mathbb{C}^d};$
- (ii): $\|\iota_{(p,d)}\|^p := \inf \left\{ C : \int_{\mathbb{B}^n} \|M_\mu^{1/p}(z)f(z)\|_{\mathbb{C}^d}^p \leq C \|f\|_{A_\alpha^p}^p \right\};$
- (iii): $\|\mu\|_{GEO} := \sup_{\lambda \in \mathbb{B}^n} \int_{D(\lambda,r)} \|M_\mu(z)\|_{\mathbb{C}^d} d\tau_\mu(z) (1 - |\lambda|^2)^{-(n+1+\alpha)};$
- (iv): $B = \sup_{\lambda_k \in \mathbb{B}^n} \int_{D(\lambda_k,r)} \|M_\mu(z)\|_{\mathbb{C}^d} d\tau_\mu(z) (1 - |\lambda_k|^2)^{-(n+1+\alpha)}$ where $\{\lambda_k\}_{k=1}^\infty$ is the sequence from Proposition 2.6;
- (v): $\|T_\mu\|_{\mathcal{L}(A_\alpha^p)}.$

Lemma 2.10. *Let $A \in M^{d \times d}$ be PSD, let $\|\cdot\|$ be any matrix norm (see Section 5.6 of [11]), and let $\lambda_1(A)$ denote the largest eigenvalue of A . Then there holds $\text{tr}(A) \simeq \|A\| \simeq \lambda_1(A)$ with implied constant depending only on d .*

Proof. Recall that all norms on $M^{d \times d}$ are equivalent with constants depending only on d . We therefore need only show that $\text{tr}(A) \simeq \|A\|_F$ where $\|A\|_F = \sqrt{\text{tr}(A^*A)}$ (i.e., it is the Frobenius Norm or Hilbert-Schmidt Norm). Let $\{\lambda_i\}_{i=1}^d$ be the eigenvalues of A arranged in decreasing order and note that $A^*A = A^2$. Then $\text{tr}(A^*A) = \sum_{i=1}^d \lambda_i^2 = (\lambda_1^2) \sum_{i=1}^d (\lambda_i/\lambda_1)^2 \leq d\lambda_1^2 \leq d \left(\sum_{i=1}^d \lambda_i \right)^2 = d\text{tr}(A)^2$. Also, $\text{tr}(A)^2 = \left(\sum_{i=1}^n \lambda_i \right)^2 \leq 2^{n-1} \sum_{i=1}^n \lambda_i^2 = 2^{n-1} \text{tr}(A^*A)$. Finally, $\lambda_1(A) \leq \text{tr}(A) \leq d\lambda_1(A)$. \square

The following is used in the next two lemmas.

Lemma 2.11. $\sup_{e \in \mathbb{C}^d, \|e\|_2=1} \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \langle d\mu(z)e, e \rangle \simeq \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 d\tau_\mu(z).$

Proof. The proof is a simple calculation that uses Lemma 2.10. Indeed,

$$\begin{aligned}
 \sup_{e \in \mathbb{C}^d, \|e\|_{\mathbb{C}^d}=1} \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} \left| k_\lambda^{(2,\alpha)}(z) \right|^2 \langle d\mu(z)e, e \rangle_{\mathbb{C}^d} &= \sup_{e \in \mathbb{C}^d, \|e\|_{\mathbb{C}^d}=1} \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} \left| k_\lambda^{(2,\alpha)}(z) \right|^2 \langle M_\mu(z)e, e \rangle_{\mathbb{C}^d} d\tau_\mu(z) \\
 &\leq \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} \left| k_\lambda^{(2,\alpha)}(z) \right|^2 \|M_\mu(z)\|_{\mathbb{C}^d} d\tau_\mu(z) \\
 &\simeq \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} \left| k_\lambda^{(2,\alpha)}(z) \right|^2 \sum_{i=1}^d \langle M_\mu(z)e_i, e_i \rangle_{\mathbb{C}^d} d\tau_\mu(z) \\
 &\simeq \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} \left| k_\lambda^{(2,\alpha)}(z) \right|^2 \sum_{i=1}^d \langle d\mu(z)e_i, e_i \rangle_{\mathbb{C}^d} \\
 &\simeq \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} \left| k_\lambda^{(2,\alpha)}(z) \right|^2 d\tau_\mu(z).
 \end{aligned}$$

This gives one of the required inequalities. For the next inequality there holds:

$$\begin{aligned}
 \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 d\tau_\mu(z) &= \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \sum_{j=1}^d \langle d\mu(z)e_j, e_j \rangle \\
 &= \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \sum_{j=1}^d \langle M_\mu(z)e_j, e_j \rangle d\tau_\mu(z)
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^d \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \langle d\mu(z) e_j, e_j \rangle \\
&\leq d \sup_{\|e\|_2=1} \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \langle d\mu(z) e, e \rangle.
\end{aligned}$$

This completes the proof. \square

Remark 2.12. Note that the lemma was stated using the function $|k_\lambda^{(2,\alpha)}|$ (because this is what will be needed), but it is true for any non-negative function.

We are now ready to prove Lemma 2.9. (The proof is simply an appropriate adaptation of the proofs given in, for example, [18, 29, 30]).

Proof. $\|\mu\|_{GEO} \simeq \|\mu\|_{RKM}$.

We will use Proposition 2.10, Proposition 2.1, and Lemma 2.11. Then,

$$\begin{aligned}
\sup_{\lambda \in \mathbb{B}^n} \frac{\int_{D(\lambda,r)} \|M_\mu(z)\| d\tau_\mu(z)}{(1 - |\lambda|^2)^{n+1+\alpha}} &= \sup_{\lambda \in \mathbb{B}^n} \int_{D(\lambda,r)} \frac{(1 - |\lambda|^2)^{n+1+\alpha}}{(1 - |\lambda|^2)^{2(n+1+\alpha)}} \|M_\mu(z)\| d\tau_\mu(z) \\
&\simeq \sup_{\lambda \in \mathbb{B}^n} \int_{D(\lambda,r)} \frac{(1 - |\lambda|^2)^{n+1+\alpha}}{|1 - \lambda \bar{z}|^{2(n+1+\alpha)}} \sum_{k=1}^d \langle M_\mu(z) e_k, e_k \rangle_{\mathbb{C}^d} d\tau_\mu(z) \\
&= \sup_{\lambda \in \mathbb{B}^n} \int_{D(\lambda,r)} |k_\lambda^{(2,\alpha)}(z)|^2 \text{tr}(d\mu(z)) \\
&\simeq \sup_{\|e\|_2=1} \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \langle d\mu(z) e, e \rangle.
\end{aligned}$$

\square

Proof. $\|T_\mu\|_{\mathcal{L}(A_\alpha^p)} \lesssim \|\iota_{(p,d)}\|^p$. Let $f, g \in H^\infty(\mathbb{B}^n; \mathbb{C}^d)$ ($H^\infty(\mathbb{B}^n; \mathbb{C}^d)$ is simply the space of bounded holomorphic \mathbb{C}^d -valued functions on \mathbb{B}^n). Then by Fubini's Theorem and Hölder's Inequality (Proposition 2.8):

$$\begin{aligned}
|\langle T_\mu f, g \rangle_{A_\alpha^2}| &= \left| \int_{\mathbb{B}^n} \left\langle \int_{\mathbb{B}^n} \frac{d\mu(w) f(w)}{(1 - \bar{w}z)^{n+1+\alpha}}, g(z) \right\rangle_{\mathbb{C}^d} dv_\alpha(z) \right| \\
&= \left| \int_{\mathbb{B}^n} \left\langle d\mu(w) f(w), \int_{\mathbb{B}^n} \frac{g(z)}{(1 - w\bar{z})^{n+1+\alpha}} dv_\alpha(z) \right\rangle_{\mathbb{C}^d} \right| \\
&\leq \int_{\mathbb{B}^n} |\langle M_\mu(w) f(w), g(w) \rangle_{\mathbb{C}^d}| d\tau_\mu(w) \\
&\leq \|f\|_{L^p(\mathbb{B}^n; \mathbb{C}^d, \mu)} \|g\|_{L^q(\mathbb{B}^n; \mathbb{C}^d, \mu)} \\
&\leq \|\iota_{(p,d)}\| \|\iota_{(q,d)}\| \|g\|_{A_\alpha^2} \|f\|_{A_\alpha^2}.
\end{aligned}$$

\square

Proof. $B \lesssim \|\mu\|_{GEO}$. This is immediate from the definitions. \square

Proof. $\|\iota_{(p,d)}\|^p \lesssim B$. Let $\{a_k\}_{k=1}^\infty$ be the sequence from Proposition 2.6. So, there holds

$$\int_{D(\lambda_k, r)} \|M_\mu(z)\| d\tau_\mu(z) (1 - |\lambda_k|^2)^{-(n+1+\alpha)} \leq B$$

for all k . Let f be holomorphic and $D_k = D(\lambda_k, 2r)$.

$$\begin{aligned}
 \int_{\mathbb{B}^n} \|M_\mu^{1/p}(z)f(z)\|_p^p d\tau_\mu(z) &\leq \sum_{k=1}^{\infty} \int_{D_k} \|f(z)\|_p^p \|M_\mu^{1/p}(z)\|^p d\tau_\mu(z) \\
 &\simeq \sum_{k=1}^{\infty} \int_{D_k} \|f(z)\|_p^p \text{tr}(M_\mu^{1/p}(z))^p d\tau_\mu(z) \\
 &\simeq \sum_{k=1}^{\infty} \int_{D_k} \|f(z)\|_p^p \|M_\mu(z)\| d\tau_\mu(z) \\
 &\leq \sum_{k=1}^{\infty} \sup_{z \in D_k} \|f(z)\|_p^p \int_{D_k} \|M_\mu(z)\| d\tau_\mu(z) \\
 &\lesssim \sum_{k=1}^{\infty} \frac{C}{(1-|\lambda_k|^2)^{n+1+\alpha}} \int_{D_k} \|f(w)\|_p^p dv_\alpha(w) \int_{D(\lambda_k, r)} \|M_\mu(z)\| d\tau_\mu(z) \\
 &= \sum_{k=1}^{\infty} C \int_{D_k} \|f(w)\|_p^p dv_\alpha(w) \int_{D_k} \frac{\|M_\mu(z)\|}{(1-|\lambda_k|^2)^{(n+1+\alpha)}} d\tau_\mu(z) \\
 &\leq \sum_{k=1}^{\infty} CB \int_{D_k} \|f(w)\|_p^p dv_\alpha(w) \\
 &\leq CBN \|f\|_{A_\alpha^p}^p.
 \end{aligned}$$

Above we use the estimate from Proposition 2.3 and the last inequality is due to the fact that each $z \in \mathbb{B}^n$ belongs to at most N of the sets $D(\lambda_k, 2r)$. \square

Proof. $\|\mu\|_{RKM} \lesssim \|T_\mu\|_{\mathcal{L}(A_\alpha^p)}$. Assume that $T_\mu \in \mathcal{L}(A_\alpha^p)$. Then

$$\begin{aligned}
 \left\langle T_\mu(k_\lambda^{(p,\alpha)}e), k_\lambda^{(q,\alpha)}e \right\rangle_{A_\alpha^2} &= \int_{\mathbb{B}^n} \left\langle T_\mu(k_\lambda^{(p,\alpha)}e)(z), k_\lambda^{(q,\alpha)}(z)e \right\rangle_{\mathbb{C}^d} dv_\alpha(z) \\
 &= \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \left\langle \frac{d\mu(w)(1-|\lambda|^2)^{\frac{n+1+\alpha}{q}}}{((1-\bar{w}z)(1-\bar{\lambda}w))^{n+1+\alpha}} e, \frac{(1-|\lambda|^2)^{\frac{n+1+\alpha}{p}}}{(1-\bar{\lambda}z)^{n+1+\alpha}} e \right\rangle_{\mathbb{C}^d} dv_\alpha(z) \\
 &= \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \left\langle \frac{d\mu(w)(1-|\lambda|^2)^{n+1+\alpha}}{(1-\bar{\lambda}w)^{n+1+\alpha}} e, \frac{K_\lambda^\alpha(z)}{(1-\bar{z}w)^{n+1+\alpha}} e \right\rangle_{\mathbb{C}^d} dv_\alpha(z) \\
 &= \int_{\mathbb{B}^n} \left\langle \frac{d\mu(w)(1-|\lambda|^2)^{n+1+\alpha}}{(1-\bar{\lambda}w)^{n+1+\alpha}} e, \frac{1}{(1-\bar{\lambda}w)^{n+1+\alpha}} e \right\rangle_{\mathbb{C}^d} \\
 &= \int_{\mathbb{B}^n} \left\langle \frac{d\mu(w)(1-|\lambda|^2)^{n+1+\alpha}}{|1-\bar{\lambda}w|^{2(n+1+\alpha)}} e, e \right\rangle_{\mathbb{C}^d} \\
 &= \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(w)|^2 \langle d\mu(w)e, e \rangle_{\mathbb{C}^d}.
 \end{aligned}$$

This computation implies:

$$\sup_{e \in \mathbb{C}^d, \|e\|_2=1} \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \langle d\mu(z)e, e \rangle_{\mathbb{C}^d} = \sup_{e \in \mathbb{C}^d, \|e\|_2=1} \sup_{\lambda \in \mathbb{B}^n} \left\langle T_\mu(k_\lambda^{(p,\alpha)}e), k_\lambda^{(q,\alpha)}e \right\rangle_{A^2}$$

$$\begin{aligned}
&\leq \sup_{\lambda \in \mathbb{B}^n} \|T_\mu\|_{A^p \rightarrow A^p} \|k_\lambda^{(p,\alpha)}\|_{A_\alpha^p} \|k_\lambda^{(q,\alpha)}\|_{A_\alpha^q} \\
&\simeq \|T_\mu\|_{A^p \rightarrow A^p}.
\end{aligned}$$

□

Proof. $\|\mu\|_{RKM} \lesssim \|\iota_{(p,d)}\|^p$. From the inequalities we have already proven, we have $\|\mu\|_{RKM} \lesssim \|T_\mu\|_{\mathcal{L}(A_\alpha^p)} \lesssim \|\iota_{(p,d)}\|^p$. □

To state the following corollary, we first define the scalar total variation, denoted $|\mu|$, of a matrix-valued measure, μ . Let $|\mu| := \sum_{i=1}^d \sum_{j=1}^d |\mu_{(i,j)}|$, where $|\mu_{(i,j)}|$ is the total variation of the measure $\mu_{(i,j)}$. In the case that μ is a PSD matrix-valued measure, there holds: $d|\mu|(z) = \sum_{i,j} d|\mu_{i,j}|(z) = \sum_{i,j} \left| \frac{d\mu_{i,j}}{d\tau_\mu}(z) \right| d\tau_\mu(z) = \sum_{i,j} |(M_\mu)_{i,j}(z)| d\tau_\mu(z) \simeq \|M_\mu(z)\| d\tau_\mu(z)$.

To emphasize, the total variation of a matrix-valued measure is a positive *scalar-valued* measure. This differs from the definition in, for example, [23] in which the total variation of a matrix-valued measure is defined to be a PSD matrix-valued measure. But our definition is not without precedent. For example, in [8], the authors define the total variation of a *vector-valued* measure to be a positive scalar-valued measure, though their definition is different from ours. Even though our definition of the total variation of a matrix-valued measure is different than the one appearing in, for example [23] and [8], this is nonetheless a reasonable definition: If ν_1 is a complex scalar measure and ν_2 is a positive measure such that $\nu_1 \ll \nu_2$, and if $d\nu_1 = f d\nu_2$ then the total variation of ν_1 is defined by $d|\nu_1| = |f| d\nu_2$. So, in the case that μ is a PSD matrix-valued measure, we are saying that $d|\mu| = \|M_\mu\| d\tau_\mu$.

Corollary 2.13. *Let $|\mu|$ be the total variation of the PSD matrix-valued measure μ . The following quantities are equivalent.*

- (i): $\|\mu\|_{RKM} := \sup_{e \in \mathbb{C}^d, \|e\|_2=1} \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 \langle d\mu(z)e, e \rangle_{\mathbb{C}^d};$
- (ii): $\|\iota_{(p,d)}\|^p := \inf \left\{ C : \int_{\mathbb{B}^n} \|M_\mu^{1/p}(z) f(z)\|_p^p d\tau_\mu(z) \leq C \|f\|_{A_\alpha^p}^p \right\};$
- (iii): $\|\mu\|_{GEO} := \sup_{\lambda \in \mathbb{B}^n} \int_{D(\lambda,r)} \|M_\mu(z)\| d\tau_\mu(z) (1 - |\lambda|^2)^{-(n+1+\alpha)};$
- (iv): $\|T_\mu\|_{\mathcal{L}(A_\alpha^p)};$
- (v): $\| |\mu| \|_{RKM} = \sup_{\lambda \in \mathbb{B}^n} \int_{\mathbb{B}^n} |k_\lambda^{(2,\alpha)}(z)|^2 d|\mu|(z);$
- (vi): $\|\kappa_{(p,d)}\|^p := \inf \left\{ C : \int_{\mathbb{B}^n} \|f(z)\|_p^p d|\mu|(z) \leq C \|f\|_{A_\alpha^p}^p \right\};$
- (vii): $\| |\mu| \|_{GEO} = \sup_{\lambda \in \mathbb{B}^n} \int_{D(\lambda,r)} d|\mu|(z) (1 - |\lambda|^2)^{-(n+1+\alpha)};$
- (viii): $\|T_{|\mu|}\|_{\mathcal{L}(A_\alpha^p)}.$

Proof. The equivalence between (i)–(iv) was proven in Lemma 2.9, and the equivalence of (v)–(viii) is well-known (see for example [18] or [30]). To prove the current theorem, we only need to “connect” the two sets of equivalencies. But this is easy since the quantities defined in (iii) and (vii) are equivalent. □

Corollary 2.14. *If μ is a Carleson matrix-valued measure or if $|\mu|$ is A_α^p -Carleson, then the variation of every entry of μ is Carleson.*

Proof. There holds:

$$\int_{\mathbb{B}^n} \|f(z)\|_p^p d|\mu_{(i,j)}|(z) = \int_{\mathbb{B}^n} \|f(z)\|_p^p |(M_\mu)_{(i,j)}(z)| d\tau_\mu(z)$$

$$\begin{aligned} &\leq \sum_{i=1}^d \sum_{j=1}^d \int_{\mathbb{B}^n} \|f(z)\|_p^p |M_{(i,j)}(z)| d\tau_\mu(z). \\ &\simeq \int_{\mathbb{B}^n} \|f(z)\|_p^p \|M_\mu(z)\| d\tau_\mu(z). \end{aligned}$$

Using Corollary 2.13,

$$\begin{aligned} \int_{\mathbb{B}^n} \|f(z)\|_p^p \|M_\mu(z)\| d\tau_\mu(z) &\simeq \int_{\mathbb{B}^n} \|f(z)\|_p^p d|\mu|(z) \\ &\leq \|T_{|\mu|}\|_{\mathcal{L}(A_\alpha^p)} \int_{\mathbb{B}^n} \|f(z)\|_p^p dv_\alpha(z). \end{aligned}$$

□

Lemma 2.15. *Let $1 < p < \infty$ and suppose that μ is an A_α^p matrix-valued Carleson measure. Let $F \subset \mathbb{B}^n$ be a compact set, then*

$$\|T_{\mu 1_F} f\|_{A_\alpha^p} \lesssim \|T_\mu\|_{\mathcal{L}(A_\alpha^p)}^{\frac{1}{q}} \|1_F f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)},$$

where $q = \frac{p}{p-1}$.

Proof. It is clear $T_{\mu 1_F} f$ is a bounded analytic function for any $f \in A_\alpha^p$ since F is compact and μ is a finite measure. As in the proof of the previous lemma, there holds

$$\begin{aligned} \left| \langle T_{\mu 1_F} f, g \rangle_{A_\alpha^2} \right| &= \left| \int_{\mathbb{B}^n} \langle M_\mu(w) 1_F(w) f(w), g(w) \rangle_{\mathbb{C}^d} d\tau_\mu(w) \right| \\ &= \langle 1_F f, g \rangle_{L^2(\mathbb{B}^n, \mathbb{C}^d; \mu)} \\ &\leq \|1_F f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)} \|g\|_{L^q(\mathbb{B}^n, \mathbb{C}^d; \mu)} \\ &\lesssim \|T_\mu\|_{\mathcal{L}(A_\alpha^p)}^{\frac{1}{q}} \|1_F f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)} \|g\|_{A_\alpha^q}. \end{aligned}$$

Note that in the above we used Proposition 2.8. □

For a Carleson measure μ and $1 < p < \infty$ and for $f \in L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)$ we also define

$$P_\mu f(z) := \int_{\mathbb{B}^n} \frac{d\mu(w) f(w)}{(1 - \bar{w}z)^{n+1+\alpha}}.$$

It is easy to see based on the computations above that P_μ is a bounded operator from $L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)$ to A_α^p and $T_\mu = P_\mu \circ \iota_p$.

3. APPROXIMATION BY LOCALIZED COMPACT OPERATORS

In this section, we will show that every operator in the Toeplitz algebra can be approximated by sums of localized compact operators. Along with some other estimates, this will help us approximate the essential norm of operators in the Toeplitz algebra. In particular, the goal of this section will be to prove the following Theorem:

Theorem 3.1. *Let $S \in \mathcal{T}_{p,\alpha}$, μ be a A_α^p matrix-valued Carleson measure and $\epsilon > 0$. Then there are Borel sets $F_j \subset G_j \subset \mathbb{B}^n$ such that*

- (i): $\mathbb{B}^n = \cup_{j=1}^\infty F_j$;
- (ii): $F_j \cap F_k = \emptyset$ if $j \neq k$;
- (iii): each point of \mathbb{B}^n lies in no more than $N = N(n)$ of the sets G_j ;

(iv): $\text{diam}_\beta G_j \leq \mathfrak{d}(p, S, \epsilon)$ for all j , and

$$\left\| ST_\mu - \sum_{j=1}^{\infty} M_{1_{F_j}} ST_{\mu 1_{G_j}} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} < \epsilon.$$

To prove this, we prove several estimates and put them together at the end of this section to prove Theorem 3.1.

Lemma 3.2. *Let $1 < p < \infty$, $\alpha > -1$, and μ be a matrix-valued Carleson measure. Suppose that $F_j, K_j \subset \mathbb{B}^n$ are Borel sets such that $\{F_j\}_{j=1}^\infty$ are pairwise disjoint and $\beta(F_j, K_j) > \sigma \geq 1$ for all j . If $0 < \gamma < \min \left\{ \frac{1}{p(n+1+\alpha)}, \frac{p-1}{p} \right\}$, then*

$$\int_{\mathbb{B}^n} \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{K_j}(w) \frac{(1-|w|^2)^{-1/p}}{|1-\bar{z}w|^{n+1+\alpha}} d|\mu|(w) \lesssim \|T_\mu\|_{\mathcal{L}(A_\alpha^p)} (1-\delta^{2n})^\gamma (1-|z|^2)^{1/p}.$$

Proof. This is a consequence of [18, Lemma 3.3], and Corollary 2.13. \square

Lemma 3.3. *Let $1 < p < \infty$ and μ be a matrix-valued A_α^p Carleson measure. Suppose that $F_j, K_j \subset \mathbb{B}^n$ are Borel sets, $a_j \in L_{M_d}^\infty$, and $b_j \in L_{M_d}^\infty(\tau_\mu)$.*

- (i): $\beta(F_j, K_j) \geq \sigma \geq 1$;
- (ii): $\text{supp } a_j \subset F_j$ and $\text{supp } b_j \subset K_j$;
- (iii): Every $z \in \mathbb{B}^n$ belongs to at most N of the sets F_j .

Then $\sum_{j=1}^\infty M_{a_j} P_\mu M_{b_j}$ is a bounded operator from A_α^p to L_α^p and there is a function $\beta_{p,\alpha}(\sigma) \rightarrow 0$ when $\sigma \rightarrow \infty$ such that:

$$\left\| \sum_{j=1}^{\infty} M_{a_j} P_\mu M_{b_j} f \right\|_{L_\alpha^p} \leq N \beta_{p,\alpha}(\sigma) \|T_\mu\|_{\mathcal{L}(A_\alpha^p)} \|f\|_{A_\alpha^p}, \quad (4)$$

and for every $f \in A_\alpha^p$

$$\sum_{j=1}^{\infty} \|M_{a_j} P_\mu M_{b_j} f\|_{L_\alpha^p}^p \leq N \beta_{p,\alpha}^p(\sigma) \|T_\mu\|_{\mathcal{L}(A_\alpha^p)}^p \|f\|_{A_\alpha^p}^p. \quad (5)$$

Proof. Since μ is a matrix-valued Carleson measure for A_α^p , $\kappa_{(p,d)}$ is bounded, with $\kappa_{(p,d)} \simeq \|\mu\|_{\text{RKM}}^{\frac{1}{p}} \simeq \|T_\mu\|_{\mathcal{L}(A_\alpha^p)} \simeq \|T_{|\mu|}\|_{\mathcal{L}(A_\alpha^p)}$ it is enough to prove the following two estimates:

$$\left\| \sum_{j=1}^{\infty} M_{a_j} P_\mu M_{b_j} f \right\|_{L_\alpha^p} \leq N \psi_{p,\alpha}(\delta) \|T_\mu\|_{\mathcal{L}(A_\alpha^p)}^{1-\frac{1}{p}} \|f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; |\mu|)}, \quad (6)$$

and

$$\sum_{j=1}^{\infty} \|M_{a_j} P_\mu M_{b_j} f\|_{L_\alpha^p}^p \leq N \psi_{p,\alpha}^p(\delta) \|T_\mu\|_{\mathcal{L}(A_\alpha^p)}^{p-1} \|f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; |\mu|)}^p \quad (7)$$

where $\delta = \tanh \frac{\sigma}{2}$ and $\psi_{p,\alpha}(\delta) \rightarrow 0$ as $\delta \rightarrow 1$. Estimates (6) and (7) imply (4) and (5) via an application of the matrix-valued Carleson Embedding Theorem, Corollary 2.13.

First, consider the case when $N = 1$, and so the sets $\{F_j\}_{j=1}^\infty$ are pairwise disjoint. Set

$$\Phi(z, w) = \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{K_j}(w) \frac{1}{|1-\bar{z}w|^{n+1+\alpha}}.$$

Suppose now that $\|a_j\|_{L^\infty_{M^d}}$ and $\|b_j\|_{L^\infty_{M^d}(\tau_\mu)} \leq 1$. There holds:

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} M_{a_j} P_\mu M_{b_j} f(z) \right\|_p &= \left\| \sum_{j=1}^{\infty} a_j(z) \int_{\mathbb{B}^n} \frac{b_j(w) f(w)}{(1 - \bar{w}z)^{n+1+\alpha}} d\mu(w) \right\|_p \\ &= \left\| \sum_{j=1}^{\infty} a_j(z) \int_{\mathbb{B}^n} \frac{M_\mu(w) b_j(w) f(w)}{(1 - \bar{w}z)^{n+1+\alpha}} d\tau_\mu(w) \right\|_p \\ &\leq \int_{\mathbb{B}^n} \Phi(z, w) \|M_\mu(w)\| \|f(w)\|_p d\tau_\mu(w) \\ &\simeq \int_{\mathbb{B}^n} \Phi(z, w) \|f(w)\|_p d|\mu|(w). \end{aligned}$$

We will show that the operator with kernel $\Phi(z, w)$ is bounded from $L^p(\mathbb{B}^n, \mathbb{C}; |\mu|)$ into L^p_α with norm controlled by a constant, $C(n, \alpha, p)$ times $\psi_{p,\alpha}(\delta) \|T_\mu\|_{\mathcal{L}(A^p_\alpha)}^{1-\frac{1}{p}}$. Assuming this is true, there holds:

$$\begin{aligned} \left\| \sum_{j=1}^{\infty} M_{a_j} P_\mu M_{b_j} f \right\|_{L^p_\alpha} &= \int_{\mathbb{B}^n} \left\| \sum_{j=1}^{\infty} M_{a_j} P_\mu M_{b_j} f(z) \right\|_p^p dv_\alpha(z) \\ &\leq \int_{\mathbb{B}^n} \left(\int_{\mathbb{B}^n} \Phi(z, w) \|f(w)\|_p d|\mu|(w) \right)^p dv_\alpha(z) \\ &\leq C(n, \alpha, p) \psi_{p,\alpha}(\delta) \|T_\mu\|_{\mathcal{L}(A^p_\alpha)}^{1-\frac{1}{p}} \int_{\mathbb{B}^n} \|f(z)\|_p^p d|\mu|(z). \end{aligned}$$

We use Schur's Test to prove that this operator is bounded. Set $h(z) = (1 - |z|^2)^{-\frac{1}{pq}}$ and observe that Lemma 3.2 gives

$$\int_{\mathbb{B}^n} \Phi(z, w) h(w)^q d|\mu|(w) \lesssim \|T_\mu\|_{\mathcal{L}(A^p_\alpha)} (1 - \delta^{2n})^\gamma h(z)^q.$$

Using Lemma 2.4, there holds

$$\begin{aligned} \int_{\mathbb{B}^n} \Phi(z, w) h(z)^p dv_\alpha(z) &= \int_{\mathbb{B}^n} \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{K_j}(w) \frac{(1 - |z|^2)^{-\frac{1}{q}}}{|1 - \bar{z}w|^{n+1+\alpha}} dv_\alpha(z) \\ &\leq \int_{\mathbb{B}^n} \sum_{j=1}^{\infty} 1_{F_j}(z) 1_{K_j}(w) (1 - |w|^2)^{-\frac{1}{q}-\alpha} dv_\alpha(z) \\ &\lesssim h(w)^p. \end{aligned}$$

Therefore, Schur's Lemma says that the operator with kernel $\Phi(z, w)$ is bounded from $L^p(\mathbb{B}^n, \mathbb{C}^d; |\mu|)$ to $L^p_\alpha(\mathbb{B}^n; \mathbb{C})$ with norm controlled by a constant $C(n, \alpha, p)$ times $\psi_{p,\alpha}(\delta) \|T_\mu\|_{\mathcal{L}(A^p_\alpha)}^{1-\frac{1}{p}}$.

This gives (6) when $N = 1$. Since the sets F_j are disjoint in this case, then we also have (7) because

$$\sum_{j=1}^{\infty} \|M_{a_j} P_\mu M_{b_j} f\|_{L^p_\alpha}^p = \left\| \sum_{j=1}^{\infty} M_{a_j} P_\mu M_{b_j} f \right\|_{L^p_\alpha}^p.$$

Now suppose that $N > 1$. Let $z \in \mathbb{B}^n$ and let $S(z) = \{j : z \in F_j\}$, ordered according to the index j . Each F_j admits a disjoint decomposition $F_j = \bigcup_{k=1}^N A_j^k$ where A_j^k is the set of $z \in F_j$ such that j is the i^{th} element of $S(z)$. Then, for $1 \leq k \leq N$ the sets $\{A_j^k : j \geq 1\}$ are pairwise disjoint. Hence, we can apply the computations obtained above to conclude that

$$\begin{aligned} \sum_{j=1}^{\infty} \|M_{a_j} P_{\mu} M_{b_j} f\|_{L_{\alpha}^p}^p &= \sum_{j=1}^{\infty} \sum_{k=1}^N \left\| M_{a_j 1_{A_j^k}} P_{\mu} M_{b_j} f \right\|_{L_{\alpha}^p}^p \\ &= \sum_{k=1}^N \sum_{j=1}^{\infty} \left\| M_{a_j 1_{A_j^k}} P_{\mu} M_{b_j} f \right\|_{L_{\alpha}^p}^p \\ &\lesssim N \psi_{p,\alpha}^p(\delta) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)}^{p-1} \|f\|_{A_{\alpha}^p}^p. \end{aligned}$$

This gives (7), and (6) follows from similar computations. \square

Lemma 3.4. *Let $1 < p < \infty$ and $\sigma \geq 1$. Suppose that $a_1, \dots, a_k \in L_{M_d}^{\infty}$ with norm at most 1 and that μ is a matrix-valued Carleson measure. Consider the covering of \mathbb{B}^n given by Lemma 2.7 for these values of k and $\sigma \geq 1$. Then there is a positive constant $C(p, k, n, \alpha)$ such that:*

$$\left\| \left[\prod_{i=1}^k T_{a_i} \right] T_{\mu} - \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{\mu 1_{F_{k+1,j}}} \right] \right\|_{\mathcal{L}(A_{\alpha}^p)} \lesssim \beta_{p,\alpha}(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)}$$

where $\beta_{p,\alpha}(\sigma) \rightarrow 0$ as $\sigma \rightarrow \infty$.

Proof. First note that the quantity:

$$\left\| \left[\prod_{i=1}^k T_{a_i} \right] T_{\mu} - \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{\mu 1_{F_{k+1,j}}} \right] \right\|_{\mathcal{L}(A_{\alpha}^p)},$$

is dominated by the sum of

$$\left\| \left[\prod_{i=1}^k T_{a_i} \right] T_{\mu} - \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{a_i 1_{F_{i,j}}} \right] T_{1_{F_{k+1,j}} \mu} \right\|_{\mathcal{L}(A_{\alpha}^p, L_{\alpha}^p)} \quad (8)$$

and

$$\left\| \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{a_i} \right] T_{\mu 1_{F_{k+1,j}}} - \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{a_i 1_{F_{i,j}}} \right] T_{\mu 1_{F_{k+1,j}}} \right\|_{\mathcal{L}(A_{\alpha}^p, L_{\alpha}^p)}. \quad (9)$$

Therefore, we only need to prove that the quantities in (8) and (9) are controlled by $\beta_{p,\alpha}(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)}$.

For $0 \leq m \leq k+1$, define the operators $S_m \in \mathcal{L}(A_{\alpha}^p, L_{\alpha}^p)$ by

$$S_m = \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^m T_{1_{F_{i,j}} a_i} \prod_{i=m+1}^k T_{a_i} \right] T_{\mu}.$$

Clearly we have $S_0 = \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{a_i} \right] T_{\mu} = \left[\prod_{i=1}^k T_{a_i} \right] T_{\mu}$, with convergence in the strong operator topology. Similarly, we have

$$S_{k+1} = \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{a_i 1_{F_{i,j}}} \right] T_{\mu 1_{F_{k+1,j}}}.$$

When $0 \leq m \leq k-1$, a simple computation gives that

$$S_m - S_{m+1} = \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^m T_{1_{F_{i,j}} a_i} \right] T_{1_{F_{m+1,j}^c} a_{m+1}} \left[\prod_{i=m+2}^k T_{a_i} \right] T_{\mu}.$$

Here, of course, we should interpret this product as the identity when the lower index is greater than the upper index. Take any $f \in A_{\alpha}^p$ and apply Lemma 3.3, in particular (5), Lemma 2.7 and some obvious estimates to see that

$$\begin{aligned} \|(S_m - S_{m+1}) f\|_{L_{\alpha}^p}^p &\leq C(p)^{pm} \sum_{j=1}^{\infty} \left\| M_{1_{F_{m,j}} a_m} P_{\alpha} M_{1_{F_{m+1,j}^c} a_{m+1}} \left[\prod_{i=m+2}^k T_{a_i} \right] T_{\mu} f \right\|_{L_{\alpha}^p}^p \\ &\leq C(p)^{pm} N \beta_{p,\alpha}^p(\sigma) \left\| \left[\prod_{i=m+2}^k T_{a_i} \right] T_{\mu} f \right\|_{L_{\alpha}^p}^p \\ &\leq C(p)^{p(k-1)} N \beta_{p,\alpha}^p(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)}^p \|f\|_{A_{\alpha}^p}^p. \end{aligned}$$

Also,

$$S_k - S_{k+1} = \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{1_{F_{i,j}} a_i} \right] T_{\mu 1_{F_{k+1,j}^c}},$$

and again applying Lemma 3.3, and in particular (5), we find that

$$\|(S_k - S_{k+1}) f\|_{L_{\alpha}^p}^p \leq C_p^{pk} N \beta_{p,\alpha}^p(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)}^p \|f\|_{A_{\alpha}^p}^p.$$

Since $N = N(n)$, we have the following estimates for $0 \leq m \leq k$,

$$\|(S_m - S_{m+1}) f\|_{L_{\alpha}^p} \lesssim \beta_{p,\alpha}(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)} \|f\|_{A_{\alpha}^p}.$$

But from this it is immediate that (8) holds,

$$\|(S_0 - S_{k+1}) f\|_{L_{\alpha}^p} \leq \sum_{m=0}^k \|(S_m - S_{m+1}) f\|_{L_{\alpha}^p} \lesssim \beta_{p,\alpha}(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)} \|f\|_{A_{\alpha}^p}.$$

The idea behind (9) is similar. For $0 \leq m \leq k$, define the operator

$$\tilde{S}_m = \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^m T_{1_{F_{i,j}} a_i} \prod_{i=m+1}^k T_{a_i} \right] T_{\mu 1_{F_{k+1,j}}},$$

so we have

$$\begin{aligned} \tilde{S}_0 &= \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{a_i} \right] T_{\mu 1_{F_{k+1,j}}} \\ \tilde{S}_k &= \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^k T_{a_i 1_{F_{i,j}}} \right] T_{\mu 1_{F_{k+1,j}}}. \end{aligned}$$

When $0 \leq m \leq k-1$, a simple computation gives

$$\tilde{S}_m - \tilde{S}_{m+1} = \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \left[\prod_{i=1}^m T_{1_{F_{i,j} a_i}} \right] T_{1_{F_{m+1,j}^c} a_{m+1}} \left[\prod_{i=m+2}^k T_{a_i} \right] T_{\mu_{1_{F_{k+1,j}}}}.$$

Again, applying obvious estimates and using Lemma 3.3 one concludes that

$$\begin{aligned} \left\| \left(\tilde{S}_m - \tilde{S}_{m+1} \right) f \right\|_{L_{\alpha}^p}^p &\leq C(p)^{p(k-1)} \beta_{p,\alpha}^p(\sigma) \sum_{j=1}^{\infty} \left\| T_{\mu_{1_{F_{k+1,j}}}} f \right\|_{A_{\alpha}^p}^p \\ &\leq C(p)^{p(k-1)} \beta_{p,\alpha}^p(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)}^{\frac{p}{q}} \sum_{j=1}^{\infty} \|1_{F_{k+1,j}} f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)}^p \\ &\leq C(p)^{p(k-1)} \beta_{p,\alpha}^p(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)}^{\frac{p}{q}} \|f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)}^p \\ &\leq NC(p)^{p(k-1)} \beta_{p,\alpha}^p(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)}^{\frac{p}{q}+1} \|f\|_{A_{\alpha}^p}^p. \end{aligned}$$

Here the second inequality uses Lemma 2.15, the next inequality uses that the sets $\{F_{k+1,j}\}_{j=1}^{\infty}$ form a covering of \mathbb{B}^n with at most $N = N(n)$ overlap, and the last inequality uses Lemma 2.9. Summing up, for $0 \leq m \leq k-1$ we have

$$\left\| \left(\tilde{S}_m - \tilde{S}_{m+1} \right) f \right\|_{L_{\alpha}^p} \lesssim \beta_{p,\alpha}(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)} \|f\|_{A_{\alpha}^p},$$

which implies

$$\left\| \left(\tilde{S}_0 - \tilde{S}_k \right) f \right\|_{L_{\alpha}^p} \leq \sum_{m=0}^{k-1} \left\| \left(\tilde{S}_m - \tilde{S}_{m+1} \right) f \right\|_{L_{\alpha}^p} \lesssim \beta_{p,\alpha}(\sigma) \|T_{\mu}\|_{\mathcal{L}(A_{\alpha}^p)} \|f\|_{A_{\alpha}^p},$$

giving (9). \square

Lemma 3.5. *Let*

$$S = \sum_{i=1}^m \left[\prod_{l=1}^{k_i} T_{a_l^i} \right] T_{\mu_i}$$

where $a_j^i \in L_{M^d}^{\infty}$. Let $k = \max_{1 \leq i \leq m} \{k_i\}$ and let μ_i be matrix-valued measures such that $|\mu_i|$ are Carleson. Given $\epsilon > 0$, there is $\sigma = \sigma(S, \epsilon) \geq 1$ such that if $\{F_{i,j}\}_{j=1}^{\infty}$ and $0 \leq i \leq k+1$ are the sets given by Lemma 2.7 for these values of σ and k , then

$$\left\| S - \sum_{j=1}^{\infty} M_{1_{F_{0,j}}} \sum_{i=1}^m \left[\prod_{l=1}^{k_i} T_{a_l^i} \right] T_{\mu_i 1_{F_{k+1,j}}} \right\|_{\mathcal{L}(A_{\alpha}^p \rightarrow L_{\alpha}^p)} < \epsilon.$$

Proof. Each μ_i is a matrix-valued measure and by Corollary 2.14, the total variation of each entry of μ_i is a scalar Carleson measure. We will use this fact to prove the present claim. Indeed, we can write each μ_i as $\sum_{j=1}^d \sum_{k=1}^d \langle \mu_i e_j, e_k \rangle E_{j,k}$. We now apply the scalar-valued version of this Lemma, which is Lemma 3.5 in [18], to each $\langle \mu_i e_j, e_k \rangle E_{j,k}$. We then use linearity and the triangle inequality to conclude the result. \square

We are finally ready to prove Theorem 3.1.

Proof. If $S \in \mathcal{T}_{p,\alpha}$ then we can find a $S_0 = \sum_{i=1}^m \Pi_{l=1}^{k_i} T_{a_i}$ such that

$$\|S - S_0\|_{\mathcal{L}(A_\alpha^p)} < \epsilon. \quad (10)$$

We also know, by Lemma 3.5, we can pick $\sigma = \sigma(S_0, \epsilon)$ and sets $F_j = F_{0,j}$ and $G_j = F_{k+1,j}$ with

$$\left\| S_0 T_\mu - \sum_{j=1}^{+\infty} M_{1_{F_j}} S_0 T_{\mu 1_{G_j}} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} < \epsilon.$$

We know that (i)-(iv) of Theorem 3.1 are satisfied by Lemma 2.7. Note that by the triangle inequality, there holds:

$$\left\| S T_\mu - \sum_{j=1}^{+\infty} M_{1_{F_j}} S T_{\mu 1_{G_j}} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \leq \|S T_\mu - S_0 T_\mu\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \quad (11)$$

$$+ \left\| S_0 T_\mu - \sum_{j=1}^{+\infty} M_{1_{F_j}} S_0 T_{\mu 1_{G_j}} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \quad (12)$$

$$+ \left\| \sum_{j=1}^{+\infty} M_{1_{F_j}} S_0 T_{\mu 1_{G_j}} - \sum_{j=1}^{+\infty} M_{1_{F_j}} S_0 T_{\mu 1_{G_j}} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)}. \quad (13)$$

The first two terms are less than ϵ . To control the third term, let $f \in A_\alpha^p$ and recall that the sequence of sets $\{F_j\}_{j=1}^\infty$ is disjoint. Then we have

$$\begin{aligned} \left\| \sum_{j=1}^\infty M_{1_{F_j}} (S - S_0) T_{\mu 1_{G_j}} f \right\|_{L_\alpha^p}^p &= \sum_{j=1}^\infty \left\| M_{1_{F_j}} (S - S_0) T_{\mu 1_{G_j}} f \right\|_{L_\alpha^p}^p \\ &\leq \epsilon^p \sum_{j=1}^\infty \left\| T_{\mu 1_{G_j}} f \right\|_{A_\alpha^p}^p \\ &\leq \epsilon^p \sum_{j=1}^\infty \left\| T_{\mu 1_{G_j}} f \right\|_{A_\alpha^p}^p \\ &\leq \epsilon^p \sum_{j=1}^\infty \|1_{G_j} f\|_{A_\alpha^p}^p \\ &\leq N \epsilon^p \|T_\mu\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \|f\|_{A_\alpha^p}^p. \end{aligned}$$

Putting together this estimate and estimates (10) and (11), Theorem 3.1 is proven. \square

4. A UNIFORM ALGEBRA AND ITS MAXIMAL IDEAL SPACE

Consider the algebra \mathcal{A} of all scalar-valued bounded uniformly continuous functions from the metric space (\mathbb{B}^n, ρ) into $(\mathbb{C}, |\cdot|)$. Furthermore, let $M_\mathcal{A}$ be the maximal ideal space of \mathcal{A} . That is, $M_\mathcal{A}$ consists of the multiplicative linear functionals on \mathcal{A} . In [18], the authors prove that if μ is a complex-valued measure whose variation is Carleson, then there is a sequence of functions $B_k(\mu) \in \mathcal{A}$ such that $T_{B_k(\mu)} \rightarrow T_\mu$ in the $\mathcal{L}(A_\alpha^p(\mathbb{B}^n; \mathbb{C}))$ norm (see also [26]). We will prove a natural generalization to the current case of matrix-valued measures. In particular, the following holds:

Theorem 4.1. *Let $1 < p < \infty$, $-1 < \alpha$, and μ be a matrix-valued measure such that $|\mu|$ is A_α^p -Carleson. Then there is a sequence of matrix-valued measures $B_k(\mu)$ such that there holds $\langle B_k(\mu)e_i, e_j \rangle_{A_\alpha^2} \in \mathcal{A}$ and $T_{B_k(\mu)} \rightarrow T_\mu$ in $\mathcal{L}(A_\alpha^p)$ norm.*

Remark 4.2. The condition $\langle B_k(\mu)e_i, e_j \rangle_{A_\alpha^2} \in \mathcal{A}$ means that every entry of $B_k(\mu)$ is in \mathcal{A} .

Proof. By Corollary 2.14, $|\mu_{(i,j)}|$ is a Carleson measure. By [18, Theorem 4.7] there exist functions $B_k(\mu_{(i,j)})$ in \mathcal{A} such that

$$\left\| T_{B_k(\mu_{(i,j)})} - T_{\mu_{(i,j)}} \right\|_{\mathcal{L}(A_\alpha^p(\mathbb{B}^n; \mathbb{C}))} \rightarrow 0. \quad (14)$$

Let $B_k(\mu) = \sum_{i=1}^d \sum_{j=1}^d B_k(\mu_{(i,j)}) E_{(i,j)}$. Then there holds:

$$\begin{aligned} \left\| T_{B_k(\mu)} - T_\mu \right\|_{\mathcal{L}(A_\alpha^p)} &= \left\| \sum_{i=1}^d \sum_{j=1}^d T_{B_k(\mu_{(i,j)}) E_{(i,j)}} - \sum_{i=1}^d \sum_{j=1}^d T_{\mu_{(i,j)} E_{(i,j)}} \right\|_{\mathcal{L}(A_\alpha^p)} \\ &= \left\| \sum_{i=1}^d \sum_{j=1}^d T_{B_k(\mu_{(i,j)}) E_{(i,j)}} - T_{\mu_{(i,j)} E_{(i,j)}} \right\|_{\mathcal{L}(A_\alpha^p)} \\ &\leq \sum_{i=1}^d \sum_{j=1}^d \left\| T_{B_k(\mu_{(i,j)}) E_{(i,j)}} - T_{\mu_{(i,j)} E_{(i,j)}} \right\|_{\mathcal{L}(A_\alpha^p)} \\ &= \sum_{i=1}^d \sum_{j=1}^d \left\| T_{(B_k(\mu_{(i,j)}) - \mu_{(i,j)}) E_{(i,j)}} \right\|_{\mathcal{L}(A_\alpha^p)} \\ &= \sum_{i=1}^d \sum_{j=1}^d \left\| T_{(B_k(\mu_{(i,j)}) - \mu_{(i,j)})} \right\|_{\mathcal{L}(A_\alpha^p(\mathbb{B}^n; \mathbb{C}))}. \end{aligned}$$

This quantity goes to zero as $k \rightarrow \infty$ by (14). Note that in the above we used the fact that $\left\| T_{\phi E_{(i,j)}} \right\|_{\mathcal{L}(A_\alpha^p)} = \|T_\phi\|_{\mathcal{L}(A_\alpha^p(\mathbb{B}^n; \mathbb{C}))}$, which is easy to see. Indeed, if $f \in \mathcal{L}(A_\alpha^p)$, then $T_{\phi E_{(i,j)}} f = \langle P\phi f, e_i \rangle_{\mathbb{C}^d} e_j = T_\phi(\langle f, e_i \rangle_{\mathbb{C}^d}) e_j$. \square

Let \mathcal{A}_d be the set of $d \times d$ matrices with entries in \mathcal{A} . Theorem 4.1 implies the following Theorem:

Theorem 4.3. *The Toeplitz Algebra $\mathcal{T}_{p,\alpha}$ equals the closed algebra generated by $\{T_a : a \in \mathcal{A}_d\}$.*

We collect some results about \mathcal{A} and $M_{\mathcal{A}}$. Their proofs can be found in, for example, [26] and [18].

Lemma 4.4. *Let $z, w, \xi \in \mathbb{B}^n$. Then there is a positive constant that depends only on n such that*

$$\rho(\varphi_z(\xi), \varphi_w(\xi)) \lesssim \frac{\rho(z, w)}{1 - |\xi|^2}.$$

Lemma 4.5. *Let (E, d) be a metric space and $f : \mathbb{B}^n \rightarrow E$ be a continuous map. Then f admits a continuous extension from $M_{\mathcal{A}}$ into E if and only if f is (ρ, d) uniformly continuous and $\overline{f(\mathbb{B}^n)}$ is compact.*

Lemma 4.6. *Let $\{z_\alpha\}$ be a net in \mathbb{B}^n converging to $x \in M_{\mathcal{A}}$. Then*

- (i): $a \circ \varphi_x \in \mathcal{A}$ for every $a \in \mathcal{A}$. In particular, $\varphi_x : \mathbb{B}^n \rightarrow M_{\mathcal{A}}$ is continuous;
- (ii): $a \circ \varphi_{z_\omega} \rightarrow a \circ \varphi_x$ uniformly on compact sets of \mathbb{B}^n for every $a \in \mathcal{A}$.

4.1. **Maps from $M_{\mathcal{A}}$ into $\mathcal{L}(A_\alpha^p, A_\alpha^p)$.** The following discussion is similar to the discussion in [18], and the proofs and “straightforward computations” are almost exactly like the scalar-valued versions. One important remark is that when using this strategy to prove “quantitative” facts, we implicitly use the fact that \mathbb{C}^d is a finite dimensional vector space and so we may “pull out” dimensional constants. As an example, consider the next lemma, Lemma 4.8. First, a definition:

Definition 4.7. *Define the operator, $U_z^{(p,\alpha)} : A_\alpha^p \rightarrow A_\alpha^p$, by the following formula:*

$$U_z^{(p,\alpha)} f(w) := f(\varphi_z(w)) \frac{(1 - |z|^2)^{\frac{n+1+\alpha}{p}}}{(1 - w\bar{z})^{\frac{2(n+1+\alpha)}{p}}} \quad (15)$$

where the argument of $(1 - w\bar{z})$ is used to define the root appearing above.

Lemma 4.8. *There holds:*

$$\|U_z^{(p,\alpha)} f\|_{A_\alpha^p} = \|f\|_{A_\alpha^p} \quad \forall f \in A_\alpha^p,$$

and $U_z^{(p,\alpha)} U_z^{(p,\alpha)} = Id_{A_\alpha^p}$.

Proof. We will use the change of variables formula in Lemma 2. There holds

$$\begin{aligned} \|U_z^{(p,\alpha)} f\|_{A_\alpha^p}^p &= \int_{\mathbb{B}^n} \left\| f(\varphi_z(w)) \frac{(1 - |z|^2)^{\frac{n+1+\alpha}{p}}}{(1 - w\bar{z})^{\frac{2(n+1+\alpha)}{p}}} \right\|_p^p dv_\alpha(w) \\ &= \int_{\mathbb{B}^n} \|f(\varphi_z(w))\|_p^p \left| \frac{(1 - |z|^2)^{n+1+\alpha}}{(1 - w\bar{z})^{2(n+1+\alpha)}} \right| dv_\alpha(w) \\ &= \int_{\mathbb{B}^n} \|f(\varphi_z(w))\|_p^p |k_z^{(2,\alpha)(w)}|^2 dv_\alpha(z) \\ &= \int_{\mathbb{B}^n} \|f(w)\|_p^p dv_\alpha(w). \end{aligned}$$

In the last equality, we used the change of variables formula and the fact that φ_z is an involution. \square

There are several ways to justify the change of variables used in the last lemma. First, we could use the scalar-valued change of variables formula directly by appealing to the fact that $w \mapsto \|f(w)\|_p$ is in $L_\alpha^p(\mathbb{B}^n; \mathbb{C})$. Secondly, we can use the change of variables formula for the scalar-valued case indirectly by first passing to the definition of the vector-valued integral, and then applying the change of variables on each summand in the definition. Either way works, and in what follows, the proofs for the vector-valued theorems can be proven similarly. Note that the operator $U_z^{(p,\alpha)}$ can be written in the form:

$$(U_z^{(p,\alpha)} f)(w) = \sum_{k=1}^d \langle (U_z^{(p,\alpha)} f)(w), e_k \rangle_{\mathbb{C}^d} e_k$$

$$\begin{aligned}
&= \sum_{k=1}^d \frac{(1 - |z|^2)^{\frac{n+1+\alpha}{p}}}{(1 - w\bar{z})^{\frac{2(n+1+\alpha)}{p}}} \langle f \circ \varphi(w), e_k \rangle_{\mathbb{C}^d} e_k \\
&= \sum_{k=1}^d (U_z^{(p,\alpha)} \langle f, e_k \rangle_{\mathbb{C}^d})(w) e_k.
\end{aligned}$$

In the above $\langle f, e_k \rangle_{\mathbb{C}^d}(w) = \langle f(w), e_k \rangle_{\mathbb{C}^d}$.

For a real number r , set

$$J_z^r(w) = \frac{(1 - |z|^2)^{r\frac{n+1+\alpha}{2}}}{(1 - w\bar{z})^{r(n+1+\alpha)}}.$$

Let I_d be the $d \times d$ identity matrix. Observe that

$$U_z^{(p,\alpha)} f(w) = (J_z^{\frac{2}{p}}(w) I_d) f(\varphi_z(w)) \quad \text{and} \quad U_z^{(p,\alpha)} = T_{J_z^{\frac{2}{p}-1} I_d} U_z^{(2,\alpha)} = U_z^{(2,\alpha)} T_{J_z^{1-\frac{2}{p}} I_d}.$$

So, if q is the conjugate exponent of p , we have

$$(U_z^{(q,\alpha)})^* = U_z^{(2,\alpha)} T_{J_z^{\frac{2}{q}-1} I_d} = T_{J_z^{1-\frac{2}{q}} I_d} U_z^{(2,\alpha)}.$$

Then using that $U_z^{(2,\alpha)} U_z^{(2,\alpha)} = Id_{A_\alpha^2}$ and straightforward computations, we obtain

$$(U_z^{(q,\alpha)})^* U_z^{(p,\alpha)} = T_{b_z I_d} \quad \text{and} \quad U_z^{(p,\alpha)} (U_z^{(q,\alpha)})^* = T_{b_z I_d}^{-1},$$

where

$$b_z(w) = \frac{(1 - \bar{w}z)^{(n+1+\alpha)(\frac{1}{q}-\frac{1}{p})}}{(1 - \bar{z}w)^{(n+1+\alpha)(\frac{1}{q}-\frac{1}{p})}}. \quad (16)$$

For $z \in \mathbb{B}^n$ and $S \in \mathcal{L}(A_\alpha^p)$ we then define the map

$$S_z := U_z^{(p,\alpha)} S (U_z^{(q,\alpha)})^*,$$

which induces a map $\Psi_S : \mathbb{B}^n \rightarrow \mathcal{L}(A_\alpha^p, A_\alpha^p)$ given by

$$\Psi_S(z) = S_z.$$

We now show how to extend the map Ψ_S continuously to a map from $M_{\mathcal{A}}$ to $\mathcal{L}(A_\alpha^p)$ when endowed with both the weak and strong operator topologies.

First, observe that $C(\mathbb{B}^n) \subset \mathcal{A}$ induces a natural projection $\pi : M_{\mathcal{A}} \rightarrow M_{C(\mathbb{B}^n)}$. If $x \in M_{\mathcal{A}}$, let

$$b_x(w) = \frac{(1 - \bar{w}\pi(x))^{(n+1+\alpha)(\frac{1}{q}-\frac{1}{p})}}{(1 - \overline{\pi(x)}w)^{(n+1+\alpha)(\frac{1}{q}-\frac{1}{p})}}. \quad (17)$$

So, when z_ω is a net in \mathbb{B}^n that tends to $x \in M_{\mathcal{A}}$, then $z_\omega = \pi(z_\omega) \rightarrow \pi(x)$ in the Euclidean metric, and so we have $b_{z_\omega} \rightarrow b_x$ uniformly on compact sets of \mathbb{B}^n and boundedly. Furthermore,

$$(U_z^{(q,\alpha)})^* U_z^{(p,\alpha)} = T_{b_z I_d} \rightarrow T_{b_x I_d} \quad \text{and} \quad (U_z^{(p,\alpha)})^* U_z^{(q,\alpha)} = T_{\bar{b}_z I_d} \rightarrow T_{\bar{b}_x I_d},$$

where convergence is in the strong operator topologies of $\mathcal{L}(A_\alpha^p)$ and $\mathcal{L}(A_\alpha^q)$, respectively. If $a \in \mathcal{A}$ then Lemma 4.6 implies $a \circ \varphi_{z_\omega} \rightarrow a \circ \varphi_x$ uniformly on compact sets of \mathbb{B}^n . The above discussion implies that

$$T_{(a \circ \varphi_{z_\omega}) b_{z_\omega} I_d} \rightarrow T_{(a \circ \varphi_x) b_x I_d}$$

in the strong operator topology associated with $\mathcal{L}(A_\alpha^p)$.

Recall that we have $k_z^{(p,\alpha)}(w)e = \frac{(1-|z|^2)^{\frac{n+1+\alpha}{q}}}{(1-\bar{z}w)^{n+1+\alpha}}e$, with $\|k_z^{(p,\alpha)}e\|_{A_\alpha^p} \approx 1$, and so

$$(1-|\xi|^2)^{\frac{n+1+\alpha}{p}} J_z^{\frac{2}{p}}(\xi)e = (1-|\varphi_z(\xi)|^2)^{\frac{n+1+\alpha}{p}} \frac{|1-\bar{z}\xi|^{\frac{2}{p}(n+1+\alpha)}}{(1-\xi\bar{z})^{\frac{2}{p}(n+1+\alpha)}}e = (1-|\varphi_z(\xi)|^2)^{\frac{n+1+\alpha}{p}} \lambda_{(p,\alpha)}(\xi, z)e.$$

Here the constant $\lambda_{(p,\alpha)}$ is unimodular, and

$$\begin{aligned} \left\langle f, (U_z^{(p,\alpha)})^* k_\xi^{(q,\alpha)} e \right\rangle_{A_\alpha^2} &= \left\langle U_z^{(p,\alpha)} f, k_\xi^{(q,\alpha)} e \right\rangle_{A_\alpha^2} \\ &= \left\langle J_z^{\frac{2}{p}}(f \circ \varphi_z), k_\xi^{(q,\alpha)} e \right\rangle_{A_\alpha^2} \\ &= \left\langle J_z^{\frac{2}{p}}(f \circ \varphi_z) \overline{k_\xi^{(q,\alpha)}}, e \right\rangle_{A_\alpha^2} \\ &= \left\langle J_z^{\frac{2}{p}}(\xi) f(\varphi_z(\xi)) (1-|\xi|^2)^{\frac{n+1+\alpha}{p}}, e \right\rangle_{\mathbb{C}^d} \\ &= \left\langle f(\varphi_z(\xi)) (1-|\varphi_z(\xi)|^2)^{\frac{n+1+\alpha}{p}} \lambda_{(p,\alpha)}(\xi, z), e \right\rangle_{\mathbb{C}^d} \\ &= \left\langle f, \overline{\lambda_{(p,\alpha)}(\xi, z)} k_{\varphi_z(\xi)}^{(q,\alpha)} e \right\rangle_{A_\alpha^2}. \end{aligned}$$

This computation yields

$$(U_z^{(p,\alpha)})^* k_\xi^{(q,\alpha)} e = \lambda_{(p,\alpha)}(\xi, z) k_{\varphi_z(\xi)}^{(q,\alpha)} e. \quad (18)$$

We use these computations to study the continuity of the above map as a function of z .

Lemma 4.9. *Fix $\xi \in \mathbb{B}^n$ and $e \in \mathbb{C}^d$. Then the map $z \mapsto (U_z^{(p,\alpha)})^* k_\xi^{(q,\alpha)} e$ is uniformly continuous from (\mathbb{B}^n, ρ) into $(A_\alpha^q, \|\cdot\|_{A_\alpha^q})$.*

Proof. For $z, w \in \mathbb{B}^n$, there holds

$$\left\| (U_z^{(p,\alpha)})^* k_\xi^{(q,\alpha)} e - (U_w^{(p,\alpha)})^* k_\xi^{(q,\alpha)} e \right\|_{A_\alpha^q(\mathbb{B}^n; \mathbb{C}^d)} = \left\| (U_z^{(p,\alpha)})^* k_\xi^{(q,\alpha)} - (U_w^{(p,\alpha)})^* k_\xi^{(q,\alpha)} \right\|_{A_\alpha^q}.$$

And now we may apply the scalar-valued version, [18, Lemma 4.8]. \square

Proposition 4.10. *Let $S \in \mathcal{L}(A_\alpha^p)$. Then the map $\Psi_S : \mathbb{B}^n \rightarrow (\mathcal{L}(A_\alpha^p), WOT)$ extends continuously to M_A .*

Proof. Bounded sets in $\mathcal{L}(A_\alpha^p)$ are metrizable and have compact closure in the weak operator topology. Since $\Psi_S(\mathbb{B}^n)$ is bounded, by Lemma 4.5, we only need to show Ψ_S is uniformly continuous from (\mathbb{B}^n, ρ) into $(\mathcal{L}(A_\alpha^p), WOT)$, where WOT is the weak operator topology. Namely, we need to demonstrate that for $f \in A_\alpha^p$ and $g \in A_\alpha^q$ the function $z \mapsto \langle S_z f, g \rangle_{A_\alpha^2}$ is uniformly continuous from (\mathbb{B}^n, ρ) into $(\mathbb{C}, |\cdot|)$.

For $z_1, z_2 \in \mathbb{B}^n$ we have

$$\begin{aligned} S_{z_1} - S_{z_2} &= U_{z_1}^{(p,\alpha)} S (U_{z_1}^{(q,\alpha)})^* - U_{z_2}^{(p,\alpha)} S (U_{z_2}^{(q,\alpha)})^* \\ &= U_{z_1}^{(p,\alpha)} S [(U_{z_1}^{(q,\alpha)})^* - (U_{z_2}^{(q,\alpha)})^*] + (U_{z_1}^{(p,\alpha)} - U_{z_2}^{(p,\alpha)}) S (U_{z_2}^{(q,\alpha)})^* \\ &= A + B. \end{aligned}$$

The terms A and B have a certain symmetry, and so it is enough to deal with either, since the argument will work in the other case as well. Observe that

$$\begin{aligned} \left| \langle Af, g \rangle_{A_\alpha^2} \right| &\leq \|U_{z_1}^{(p,\alpha)} S\|_{\mathcal{L}(A_\alpha^p)} \|[(U_{z_1}^{(q,\alpha)})^* - (U_{z_2}^{(q,\alpha)})^*]f\|_{A_\alpha^q} \|g\|_{A_\alpha^q} \\ \left| \langle Bf, g \rangle_{A_\alpha^2} \right| &\leq \|(U_{z_1}^{(q,\alpha)})^* S\|_{\mathcal{L}(A_\alpha^p)} \|[(U_{z_1}^{(p,\alpha)})^* - (U_{z_2}^{(p,\alpha)})^*]g\|_{A_\alpha^q} \|f\|_{A_\alpha^p}. \end{aligned}$$

Since S is bounded and since $\|U_z^{(p,\alpha)}\|_{\mathcal{L}(A_\alpha^p)} \leq C(p, \alpha)$ for all z , we just need to show the expression

$$\|[(U_{z_1}^{(p,\alpha)})^* - (U_{z_2}^{(p,\alpha)})^*]g\|_{A_\alpha^q}$$

can be made small. It suffices to do this on a dense set of functions, and in particular we can take the linear span of $\{k_w^{(p,\alpha)} e : w \in \mathbb{B}^n; e \in \mathbb{C}^d\}$. Then we can apply Lemma 4.9 to conclude the result. \square

This proposition allows us to define S_x for $x \in M_A \setminus \mathbb{B}^n$. We set $S_x := \Psi_S(x)$. If (z_ω) is a net that converges to $x \in M_\alpha$, then $S_{z_\omega} \rightarrow S_x$ in WOT. In Proposition 4.12, we will show that if $S \in \mathcal{T}_{p,\alpha}$, then this convergence also takes place in SOT.

Lemma 4.11. *If (z_ω) is a net in \mathbb{B}^n converging to $x \in M_A$, then $T_{b_x I_d}$ is invertible and $T_{b_{z_\omega} I_d}^{-1} \rightarrow T_{b_x I_d}^{-1}$ in the strong operator topology.*

Proof. By Proposition 4.10 applied to the operator $S = Id_{A_\alpha^p}$ we have that $U_{z_\omega}^{(p,\alpha)} (U_{z_\omega}^{(q,\alpha)})^* = T_{b_{z_\omega} I_d}^{-1}$ has a weak operator limit in $\mathcal{L}(A_\alpha^p)$, denote this by Q . The Uniform Boundedness Principle then says that there is a constant C such that $\|T_{b_{z_\omega} I_d}^{-1}\|_{\mathcal{L}(A_\alpha^p)} \leq C$ for all ω . Then, given $f \in A_\alpha^p$ and $g \in A_\alpha^q$, since we know that

$$\left\| \left(T_{b_{z_\omega} I_d} - T_{b_x I_d} \right) g \right\|_{A_\alpha^q} \rightarrow 0,$$

there holds

$$\begin{aligned} \langle T_{b_x I_d} Qf, g \rangle_{A_\alpha^2} &= \langle Qf, T_{b_x I_d} g \rangle_{L_\alpha^2} \\ &= \lim_\omega \left\langle T_{b_{z_\omega} I_d}^{-1} f, T_{b_x I_d} g \right\rangle_{L_\alpha^2} \\ &= \lim_\omega \left(\left\langle T_{b_{z_\omega} I_d}^{-1} f, \left(T_{b_x I_d} - T_{b_{z_\omega} I_d} \right) g \right\rangle_{L_\alpha^2} + \left\langle T_{b_{z_\omega} I_d}^{-1} f, T_{b_{z_\omega} I_d} g \right\rangle_{L_\alpha^2} \right) \\ &= \langle f, g \rangle_{L_\alpha^2} + \lim_\omega \left\langle T_{b_{z_\omega} I_d}^{-1} f, \left(T_{b_x I_d} - T_{b_{z_\omega} I_d} \right) g \right\rangle_{L_\alpha^2} \\ &= \langle f, g \rangle_{L_\alpha^2}. \end{aligned}$$

This gives $T_{b_x I_d} Q = Id_{A_\alpha^p}$. Since taking adjoints is a continuous operation in the WOT, $T_{b_{z_\omega} I_d}^{-1} \rightarrow Q^*$, and interchanging the roles of p and q , we have $T_{b_x I_d} Q^* = Id_{A_\alpha^q}$, which implies that $QT_{b_x I_d} = Id_{A_\alpha^p}$. So, $Q = T_{b_x I_d}^{-1}$ and $T_{b_{z_\omega} I_d}^{-1} \rightarrow T_{b_x I_d}^{-1}$ in the weak operator topology. Finally,

$$T_{b_{z_\omega} I_d}^{-1} - T_{b_x I_d}^{-1} = T_{b_{z_\omega} I_d}^{-1} (T_{b_x I_d} - T_{b_{z_\omega} I_d}) T_{b_x I_d}^{-1},$$

and since $\|T_{b_{z_\omega} I_d}^{-1}\|_{\mathcal{L}(A_\alpha^p)} \leq C$ and $T_{b_{z_\omega} I_d} - T_{b_x I_d} \rightarrow 0$ in the strong operator topology, we also have $T_{b_{z_\omega} I_d}^{-1} \rightarrow T_{b_x I_d}^{-1}$ in the strong operator topology as claimed. \square

Proposition 4.12. *If $S \in \mathcal{T}_{p,\alpha}$ and (z_ω) is a net in \mathbb{B}^n that tends to $x \in M_{\mathcal{A}}$, then $S_{z_\omega} \rightarrow S_x$ in the strong operator topology. In particular, $\Psi_S : \mathbb{B}^n \rightarrow (\mathcal{L}(A_\alpha^p), SOT)$ extends continuously to $M_{\mathcal{A}}$.*

Proof. First observe that if $A, B \in \mathcal{L}(A_\alpha^p)$ then

$$\begin{aligned} (AB)_z &= U_z^{(p,\alpha)} AB (U_z^{(q,\alpha)})^* \\ &= U_z^{(p,\alpha)} A (U_z^{(q,\alpha)})^* (U_z^{(q,\alpha)})^* U_z^{(p,\alpha)} U_z^{(p,\alpha)} B (U_z^{(q,\alpha)})^* \\ &= A_z T_{b_z I_d} B_z. \end{aligned}$$

In general, this applies to longer products of operators.

For $S \in \mathcal{T}_{p,\alpha}$ and $\epsilon > 0$, by Theorem 4.1 there is a finite sum of finite products of Toeplitz operators with symbols in \mathcal{A} such that $\|R - S\|_{\mathcal{L}(A_\alpha^p)} < \epsilon$, and so $\|R_z - S_z\|_{\mathcal{L}(A_\alpha^p)} < C(p, \alpha)\epsilon$. Passing to the *WOT* limit we have $\|R_x - S_x\|_{\mathcal{L}(A_\alpha^p)} < C(p, \alpha)\epsilon$ for all $x \in M_{\mathcal{A}}$. These observations imply that it suffices to prove the Lemma for R alone, and then by linearity, it suffices to consider the special case $R = \prod_{j=1}^m T_{a_j E_{i,k}}$, where $a_j \in \mathcal{A}$. Recall that $E_{i,k}$ is the $d \times d$ matrix with a 1 in the (i, k) position and zeros everywhere else. A simple computation shows that

$$U_z^{(2,\alpha)} T_a U_z^{(2,\alpha)} = T_{a \circ \varphi_z}$$

and more generally,

$$\begin{aligned} (T_a)_z &= U_z^{(p,\alpha)} (U_z^{(q,\alpha)})^* (U_z^{(q,\alpha)})^* T_a U_z^{(p,\alpha)} U_z^{(p,\alpha)} (U_z^{(q,\alpha)})^* \\ &= U_z^{(p,\alpha)} (U_z^{(q,\alpha)})^* \left(T_{J_z^{-1-\frac{2}{q}}} U_z^{(2,\alpha)} T_a U_z^{(2,\alpha)} T_{J_z^{1-\frac{2}{p}}} \right) U_z^{(p,\alpha)} (U_z^{(q,\alpha)})^* \\ &= T_{b_z}^{-1} T_{(a \circ \varphi_z) b_z} T_{b_z}^{-1}. \end{aligned}$$

We now combine this computation with the observation at the beginning of the proposition to see that

$$\begin{aligned} \left(\prod_{j=1}^m T_{a_j} \right)_z &= (T_{a_1})_z T_{b_z} \cdots T_{b_z} (T_{a_m})_z \\ &= T_{b_z}^{-1} T_{(a_1 \circ \varphi_z) b_z} T_{b_z}^{-1} T_{(a_2 \circ \varphi_z) b_z} T_{b_z}^{-1} \cdots T_{b_z}^{-1} T_{(a_m \circ \varphi_z) b_z} T_{b_z}^{-1}. \end{aligned}$$

But, since the product of *SOT* nets is *SOT* convergent, Lemma 4.11 and the fact that $T_{(a \circ \varphi_{z_\omega}) b_{z_\omega}} \rightarrow T_{(a \circ \varphi_x) b_x}$ in the *SOT*, give

$$\left(\prod_{j=1}^m T_{a_j} \right)_{z_\alpha} \rightarrow T_{b_x}^{-1} T_{(a_1 \circ \varphi_x) b_x} T_{b_x}^{-1} T_{(a_2 \circ \varphi_x) b_x} T_{b_x}^{-1} \cdots T_{b_x}^{-1} T_{(a_m \circ \varphi_x) b_x} T_{b_x}^{-1}.$$

But this is exactly the statement $R_{z_\omega} \rightarrow R_x$ in the *SOT* for the operator $\prod_{j=1}^m T_{a_j}$, and proves the claimed continuous extension. \square

Before continuing, we prove that the Berezin transform is one-to-one. The following proof is an adaptation of the corresponding scalar-valued proof found in [30, Proposition 6.2].

Lemma 4.13. *The Berezin transform is one to one. That is, if $\tilde{T} = 0$, then $T = 0$.*

Proof. Let $T \in \mathcal{L}(A_\alpha^p)$ and suppose that $\tilde{T} = 0_d$. (Here, of course, 0_d is the $d \times d$ zero matrix.) Then there holds:

$$0 = \langle T(k_z^{(p,\alpha)} e_i), k_z^{(q,\alpha)} e_k \rangle_{A_\alpha^2} = \frac{1}{K^{(\alpha)}(z, z)} \langle T(K_z^{(\alpha)} e_i), K_z^{(\alpha)} e_k \rangle_{A_\alpha^2}$$

for all $z \in \mathbb{B}^n$ and for all $1 \leq i, k \leq d$. In particular, there holds:

$$\frac{1}{K^{(\alpha)}(z, z)} \langle T(K_z^{(\alpha)} e_i), K_z^{(\alpha)} e_k \rangle_{A_\alpha^2} \equiv 0.$$

Consider the function

$$F(z, w) = \langle T(K_w^{(\alpha)} e_i), K_z^{(\alpha)} e_k \rangle_{A_\alpha^2}.$$

This function is analytic in z , conjugate analytic in w and $F(z, z) = 0$ for all $z \in \mathbb{B}^n$. By a standard results for several complex variables (see for instance [14, Exercise 3 pg 365]) this implies that F is identically 0. Using the reproducing property, we conclude that

$$F(z, w) = \langle T(K_w^{(\alpha)} e_i)(z), e_k \rangle_{\mathbb{C}^d} \equiv 0,$$

and hence

$$T(K_w^{(\alpha)} e_i)(z) \equiv 0,$$

for every $w \in \mathbb{B}^n$ and $1 \leq i \leq d$. Since the products $K_w e_i$ span A_α^p , we conclude that $T \equiv 0$ and the desired result follows. \square

Proposition 4.14. *Let $S \in \mathcal{L}(A_\alpha^p)$. Then $B(S)(z) \rightarrow 0$ as $|z| \rightarrow 1$ if and only if $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$.*

Proof. If $z, \xi \in \mathbb{B}^n$, then we have

$$\begin{aligned} \langle B(S_z)(\xi) e_i, e_j \rangle_{\mathbb{C}^d} &= \left\langle S (U^{(q,\alpha)})^* k_\xi^{(p,\alpha)} e_i, (U^{(p,\alpha)})^* k_\xi^{(q,\alpha)} e_j \right\rangle_{A_\alpha^2} \\ &= \lambda_{(q,\alpha)}(\xi, z) \overline{\lambda_{(p,\alpha)}(\xi, z)} \left\langle S k_{\varphi_z(\xi)}^{(p,\alpha)} e_i, k_{\varphi_z(\xi)}^{(q,\alpha)} e_j \right\rangle_{A_\alpha^2}. \end{aligned}$$

Thus, $|\langle B(S_z)(\xi) e_i, e_j \rangle_{\mathbb{C}^d}| = |\langle B(S)(\varphi_z(\xi)) e_i, e_j \rangle_{\mathbb{C}^d}|$ since $\lambda_{(p,\alpha)}$ and $\lambda_{(q,\alpha)}$ are unimodular numbers. For $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$ and $\xi \in \mathbb{B}^n$ fixed, if (z_ω) is a net in \mathbb{B}^n tending to x , the continuity of Ψ_S in the *WOT* and Proposition 4.10 give that $B(S_{z_\omega})(\xi) \rightarrow B(S_x)(\xi)$, and consequently $|\langle B(S)(\varphi_{z_\omega}(\xi)) e_i, e_j \rangle_{\mathbb{C}^d}| \rightarrow |\langle B(S_x)(\xi) e_i, e_j \rangle_{\mathbb{C}^d}|$.

Now, suppose that $B(S)(z)$ vanishes as $|z| \rightarrow 1$. Since $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$ and $z_\omega \rightarrow x$, we have that $|z_\omega| \rightarrow 1$, and similarly $|\varphi_{z_\omega}(\xi)| \rightarrow 1$. Since $B(S)(z)$ vanishes as we approach the boundary, $B(S_x)(\xi) = 0$, and since $\xi \in \mathbb{B}^n$ was arbitrary and the Berezin transform is one-to-one, we see that $S_x = 0$.

Conversely, suppose that the Berezin transform does not vanish as we approach the boundary. Then there is a sequence $\{z_k\}_{k=1}^\infty$ in \mathbb{B}^n such that $|z_k| \rightarrow 1$ and $|\langle B(S)(z_k) e_i, e_j \rangle_{\mathbb{C}^d}| \geq \delta > 0$, for all $i, j = 1, \dots, d$. Since $M_{\mathcal{A}}$ is compact, we can extract a subnet (z_ω) of $\{z_k\}_{k=1}^\infty$ converging in $M_{\mathcal{A}}$ to $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$. The computations above imply $|\langle B(S_x)(0) e_i, e_j \rangle_{\mathbb{C}^d}| \geq \delta > 0$, which gives that $S_x \neq 0$. \square

5. CHARACTERIZATION OF THE ESSENTIAL NORM ON A_α^p WEIGHTED BERGMAN SPACES

We have now collected enough tools to provide a characterization of the essential norm of an operator on A_α^p . Even more than in the previous sections, this section uses the arguments of [18] in a nearly verbatim way. Fix $\varrho > 0$ and let $\{w_m\}_{m=1}^\infty$ and $\{D_m\}_{m=1}^\infty$ be the points and sets of Lemma 2.5. Define the measure

$$\mu_\varrho := \sum_{m=1}^{\infty} v_\alpha(D_m) \delta_{w_m} I_d,$$

The measure

$$\nu_\varrho := \sum_{m=1}^{\infty} v_\alpha(D_m) \delta_{w_m}$$

is well-known to be an A_α^p Carleson measure, so the measure μ_ϱ is also Carleson. This implies that $T_{\mu_\varrho} : A_\alpha^p \rightarrow A_\alpha^p$ is bounded. The following lemma is easily deduced from [18, Lemma 5.1] (in which the authors refer the reader to [6, 16] for a proof), and we omit the proof.

Lemma 5.1. $T_{\mu_\varrho} \rightarrow Id_{A_\alpha^p}$ on $\mathcal{L}(A_\alpha^p)$ when $\varrho \rightarrow 0$.

Now choose $0 < \varrho \leq 1$ so that $\|T_{\mu_\varrho} - Id_{A_\alpha^p}\|_{\mathcal{L}(A_\alpha^p)} < \frac{1}{4}$ and consequently $\|T_{\mu_\varrho}\|_{\mathcal{L}(A_\alpha^p)}$ and $\|T_{\mu_\varrho}^{-1}\|_{\mathcal{L}(A_\alpha^p)}$ are less than $\frac{3}{2}$. Fix this value of ϱ , and denote $\mu_\varrho := \mu$ for the rest of the paper.

For $S \in \mathcal{L}(A_\alpha^p, A_\alpha^p)$ and $r > 0$, let

$$\mathbf{a}_S(r) := \overline{\lim}_{|z| \rightarrow 1} \sup \left\{ \|Sf\|_{A_\alpha^p} : f \in T_{\mu 1_{D(z,r)}}(A_\alpha^p), \|f\|_{A_\alpha^p} \leq 1 \right\},$$

and then define

$$\mathbf{a}_S := \lim_{r \rightarrow 1} \mathbf{a}_S(r).$$

Since for $r_1 < r_2$ we have $T_{\mu 1_{D(z,r_1)}}(A_\alpha^p) \subset T_{\mu 1_{D(z,r_2)}}(A_\alpha^p)$ and $\mathbf{a}_S(r) \leq \|S\|_{\mathcal{L}(A_\alpha^p)}$, this limit is well defined. We define two other measures of the size of an operator which are given in a very intrinsic and geometric way:

$$\begin{aligned} \mathbf{b}_S &:= \sup_{r>0} \overline{\lim}_{|z| \rightarrow 1} \left\| M_{1_{D(z,r)}} S \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)}, \\ \mathbf{c}_S &:= \lim_{r \rightarrow 1} \left\| M_{1_{(r\mathbb{B}^n)^c}} S \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)}. \end{aligned}$$

In the last definition, for notational simplicity, we let $(r\mathbb{B}^n)^c = \mathbb{B}^n \setminus r\mathbb{B}^n$. Finally, for $S \in \mathcal{L}(A_\alpha^p)$ recall that

$$\|S\|_e = \inf \left\{ \|S - Q\|_{\mathcal{L}(A_\alpha^p, A_\alpha^p)} : Q \text{ is compact} \right\}.$$

We first show how to compute the essential norm of an operator S in terms of the operators S_x , where $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$.

Theorem 5.2. *Let $S \in \mathcal{T}_{p,\alpha}$. Then there exists a constant $C(p, \alpha, n)$ such that*

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x\|_{\mathcal{L}(A_\alpha^p)} \lesssim \|S\|_e \lesssim \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x\|_{\mathcal{L}(A_\alpha^p)}. \quad (19)$$

Proof. For S compact, (19) is easy to demonstrate. Since $k_\xi^{(p,\alpha)} \rightarrow 0$ weakly as $|\xi| \rightarrow 1$, then $\langle k_\xi^{(p,\alpha)} e, f \rangle_{A_\alpha^2} = \sum_{i=1}^d \langle e, e_i \rangle_{\mathbb{C}^d} \langle k_\xi^{(p,\alpha)} e_i, f \rangle_{A_\alpha^2} \rightarrow 0$ for every $f \in A_\alpha^q$ and so $k_\xi^{(p,\alpha)} e \rightarrow 0$ weakly as $|\xi| \rightarrow 1$. Therefore, $\|Sk_\xi^{(p,\alpha)} e\|_{A_\alpha^p}$ goes to 0 as well. Thus, we have

$$\left| \langle \tilde{S}(\xi)e, h \rangle_{\mathbb{C}^d} \right| = \left| \langle Sk_\xi^{(p,\alpha)} e, k_\xi^{(q,\alpha)} h \rangle_{A_\alpha^2} \right| \leq \|Sk_\xi^{(p,\alpha)} e\|_{A_\alpha^p} \|k_\xi^{(q,\alpha)} h\|_{A_\alpha^q} \approx \|Sk_\xi^{(p,\alpha)} e\|_{A_\alpha^p}. \quad (20)$$

Hence, the compactness of S implies that the Berezin transform vanishes as $|\xi| \rightarrow 1$. Then Proposition 4.14 gives that $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$.

Now let S be any bounded operator on A_α^p and suppose that Q is a compact operator on A_α^p . Select $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$ and a net (z_ω) in \mathbb{B}^n tending to x . Since the maps $U_{z_\omega}^{(p,\alpha)}$ and $U_{z_\omega}^{(q,\alpha)}$ are isometries on A_α^p and A_α^q , we have

$$\|S_{z_\omega} + Q_{z_\omega}\|_{\mathcal{L}(A_\alpha^p)} \leq \|S + Q\|_{\mathcal{L}(A_\alpha^p)}.$$

Since $S_{z_\omega} + Q_{z_\omega} \rightarrow S_x$ in the WOT , passing to the limit we get

$$\|S_x\|_{\mathcal{L}(A_\alpha^p)} \lesssim \underline{\lim} \|S_{z_\omega} + Q_{z_\omega}\|_{\mathcal{L}(A_\alpha^p)} \leq \|S + Q\|_{\mathcal{L}(A_\alpha^p)},$$

which gives

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x\|_{\mathcal{L}(A_\alpha^p)} \lesssim \|S\|_e,$$

the first inequality in (19). It only remains to address the last inequality. To accomplish this, we will instead prove that

$$\mathbf{a}_S \lesssim \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x\|_{\mathcal{L}(A_\alpha^p)}. \quad (21)$$

Then we compare this with the first inequality in (26), $\|S\|_e \lesssim \mathbf{a}_S$, shown below, to obtain

$$\|S\|_e \lesssim \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x\|_{\mathcal{L}(A_\alpha^p)}.$$

Also note that if (21) holds, then

$$\mathbf{a}_S \lesssim \|S\|_e \quad (22)$$

is also true. We now turn to addressing (21). It suffices to demonstrate that

$$\mathbf{a}_S(r) \lesssim \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x\|_{\mathcal{L}(A_\alpha^p)} \quad \forall r > 0.$$

Fix a radius $r > 0$. By the definition of $\mathbf{a}_S(r)$ there is a sequence $\{z_j\}_{j=1}^\infty \subset \mathbb{B}^n$ tending to $\partial\mathbb{B}^n$ and a normalized sequence of functions $f_j \in T_{\mu 1_{D(z_j, r)}}(A_\alpha^p)$ with $\|Sf_j\|_{A_\alpha^p} \rightarrow \mathbf{a}_S(r)$. To each f_j we have a corresponding $h_j \in A_\alpha^p$, and then

$$\begin{aligned} f_j(w) &= T_{\mu 1_{D(z_j, r)}} h_j(w) \\ &= \sum_{w_m \in D(z_j, r)} \frac{v_\alpha(D_m)}{(1 - \overline{w}_m w)^{n+1+\alpha}} h_j(w_m) \\ &= \sum_{w_m \in D(z_j, r)} \frac{(1 - |w_m|^2)^{\frac{n+1+\alpha}{q}}}{(1 - \overline{w}_m w)^{n+1+\alpha}} a_{j,m} \\ &= \sum_{w_m \in D(z_j, r)} k_{w_m}^{(p,\alpha)}(w) a_{j,m}, \end{aligned}$$

where $a_{j,m} = v_\alpha(D_m)(1 - |w_m|^2)^{-\frac{n+1+\alpha}{q}} h_j(w_m)$. We then have that

$$\left(U_{z_j}^{(q,\alpha)}\right)^* f_j(w) = \sum_{\varphi_{z_j}(w_m) \in D(0,r)} k_{\varphi_{z_j}(w_m)}^{(p,\alpha)}(w) a'_{j,m},$$

where $a'_{j,m}$ is simply the original constant $a_{j,m}$ multiplied by the unimodular constant $\lambda_{(q,\alpha)}$.

Observe that the points $|\varphi_{z_j}(w_m)| \leq \tanh r$. For j fixed, arrange the points $\varphi_{z_j}(w_m)$ such that $|\varphi_{z_j}(w_m)| \leq |\varphi_{z_j}(w_{m+1})|$ and $\arg \varphi_{z_j}(w_m) \leq \arg \varphi_{z_j}(w_{m+1})$. Since the Möbius map φ_{z_j} preserves the hyperbolic distance between the points $\{w_m\}_{m=1}^\infty$ we have for $m \neq k$ that

$$\beta(\varphi_{z_j}(w_m), \varphi_{z_j}(w_k)) = \beta(w_m, w_k) \geq \frac{\varrho}{4} > 0.$$

Thus, there can only be at most $N_j \leq M(\varrho, r)$ points in the collection $\varphi_{z_j}(w_m)$ belonging to the disc $D(0, z_j)$. By passing to a subsequence, we can assume that $N_j = M$ and is independent of j .

For the fixed j , and $1 \leq m \leq M$, select scalar-valued $g_{j,k} \in H^\infty$ with $\|g_{j,k}\|_{H^\infty} \leq C(\tanh r, \frac{\varrho}{4})$, such that $g_{j,k}(\varphi_{z_j}(w_m)) = \delta_{k,m}$, the Kronecker delta, when $1 \leq k \leq M$. The existence of the functions is easy to deduce from a result of Berndtsson [5], see also [26]. We then have

$$\begin{aligned} \left\langle \left(U_{z_j}^{(q,\alpha)}\right)^* f_j, g_{j,k} e \right\rangle_{A_\alpha^2} &= \int_{\mathbb{B}^n} \left\langle \left(U_{z_j}^{(q,\alpha)}\right)^* f_j(w), g_{j,k}(w) e \right\rangle_{\mathbb{C}^d} dv_\alpha(w) \\ &= \int_{\mathbb{B}^n} \sum_{\varphi_{z_j}(w_m) \in D(0,r)} k_{\varphi_{z_j}(w_m)}^{(p,\alpha)}(w) \langle a'_{j,m}, g_{j,k}(w) e \rangle_{\mathbb{C}^d} dv_\alpha(w) \\ &= \sum_{\varphi_{z_j}(w_m) \in D(0,r)} \left(1 - |\varphi_{z_j}(w_m)|^2\right)^{\frac{n+1+\alpha}{q}} \langle a'_{j,m}, g_{j,k}(\varphi_{z_j}(w_m)) e \rangle_{\mathbb{C}^d} \\ &= \left(1 - |\varphi_{z_j}(w_k)|^2\right)^{\frac{n+1+\alpha}{q}} \langle a'_{j,k}, e \rangle_{\mathbb{C}^d}. \end{aligned}$$

This expression implies that the sequence $\left|\langle a'_{j,k}, e \rangle_{\mathbb{C}^d}\right| \leq C = C(n, p, \varrho, r, \alpha)$ independently of j and k , because $g_{j,k} \in H^\infty$ has norm controlled by $C(r, \varrho)$, $\left(U_z^{(q,\alpha)}\right)^*$ is a bounded operator, and f_j is a normalized sequence of functions in A_α^p .

Now $(\varphi_{z_j}(w_1), \dots, \varphi_{z_j}(w_M), \langle a'_{j,1}, e \rangle_{\mathbb{C}^d}, \dots, \langle a'_{j,M}, e \rangle_{\mathbb{C}^d}) \in \mathbb{C}^{M(n+1)}$ is a bounded sequence in j , and passing to a subsequence if necessary, we can assume that converges to a point $(v_1, \dots, v_M, \langle a'_1, e \rangle_{\mathbb{C}^d}, \dots, \langle a'_M, e \rangle_{\mathbb{C}^d})$. Here $|v_k| \leq \tanh r$ and $|a'_k| \leq C$. This implies

$$\left\langle \left(U_{z_j}^{(q,\alpha)}\right)^* f_j, e \right\rangle_{\mathbb{C}^d} \rightarrow \sum_{k=1}^M \langle k_{v_k}^{(p,\alpha)} a'_k, e \rangle_{\mathbb{C}^d}$$

which means that:

$$\left(U_{z_j}^{(q,\alpha)}\right)^* f_j \rightarrow \sum_{k=1}^M k_{v_k}^{(p,\alpha)} a'_k =: h$$

in the L_α^p norm and moreover,

$$\left\| \sum_{k=1}^M k_{v_k}^{(p,\alpha)} a'_k \right\|_{L_\alpha^p} = \lim_{j \rightarrow \infty} \left\| \left(U_{z_j}^{(q,\alpha)} \right)^* f_j \right\|_{L_\alpha^p}.$$

Since the operator $U_{z_j}^{(p,\alpha)}$ is isometric and $\|S_{z_j}\|_{\mathcal{L}(A_\alpha^p, A_\alpha^p)}$ is bounded independently of j ,

$$\mathfrak{a}_S(r) = \lim_{j \rightarrow \infty} \|S f_j\|_{A_\alpha^p} = \lim_{j \rightarrow \infty} \|S_{z_j} (U_{z_j}^{(q,\alpha)})^* f_j\|_{A_\alpha^p} = \lim_{j \rightarrow \infty} \|S_{z_j} h\|_{A_\alpha^p}.$$

Since $|z_j| \rightarrow 1$, by using the compactness of $M_{\mathcal{A}}$ it is possible to extract a subnet (z_ω) which converges to some point $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$. Then $S_{z_\omega} h \rightarrow S_x h$ in A_α^p , so

$$\mathfrak{a}_S(r) = \lim_{\omega} \|S_{z_\omega} h\|_{A_\alpha^p} = \|S_x h\|_{A_\alpha^p} \lesssim \|S_x\|_{\mathcal{L}(A_\alpha^p)} \lesssim \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x\|_{\mathcal{L}(A_\alpha^p)}.$$

The above limit uses the continuity in the *SOT* as guaranteed by Proposition 4.12. \square

Theorem 5.3. *Let $1 < p < \infty$, $\alpha > -1$ and $S \in \mathcal{T}_{p,\alpha}$. Then there exist constants depending only on n , p , and α such that:*

$$\mathfrak{a}_S \approx \mathfrak{b}_S \approx \mathfrak{c}_S \approx \|S\|_e.$$

Proof. By Theorem 3.1 there are Borel sets $F_j \subset G_j \subset \mathbb{B}^n$ such that

- (i): $\mathbb{B}^n = \bigcup_{j=1}^\infty F_j$;
- (ii): $F_j \cap F_k = \emptyset$ if $j \neq k$;
- (iii): each point of \mathbb{B}^n lies in no more than $N(n)$ of the sets G_j ;
- (iv): $\text{diam}_\beta G_j \leq \mathfrak{d}(p, S, \epsilon)$

and

$$\left\| ST_\mu - \sum_{j=1}^\infty M_{1_{F_j}} ST_{\mu 1_{G_j}} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} < \epsilon. \quad (23)$$

Set

$$S_m = \sum_{j=m}^\infty M_{1_{F_j}} ST_{\mu 1_{G_j}}.$$

Next, we consider one more measure of the size of S ,

$$\overline{\lim}_{m \rightarrow \infty} \left\| \sum_{j=m}^\infty M_{1_{F_j}} ST_{\mu 1_{G_j}} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} = \overline{\lim}_{m \rightarrow \infty} \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)}.$$

First some observations. Since every $z \in \mathbb{B}^n$ belongs to only $N(n)$ sets G_j , Lemma 2.15 gives

$$\sum_{j=m}^\infty \left\| T_{\mu 1_{G_j}} f \right\|_{A_\alpha^p}^p \lesssim \sum_{j=1}^\infty \|1_{G_j} f\|_{L^p(\mathbb{B}^n, \mathbb{C}^d; \mu)}^p \lesssim \|f\|_{A_\alpha^p}^p. \quad (24)$$

Also, since T_μ is bounded and invertible, we have that $\|S\|_e \approx \|ST_\mu\|_e$. Finally, we will need to compute both norms in $\mathcal{L}(A_\alpha^p, A_\alpha^p)$ and $\mathcal{L}(A_\alpha^p, L_\alpha^p)$. When necessary, we will denote the respective essential norms as $\|\cdot\|_e$ and $\|\cdot\|_{ex}$. It is easy to show that

$$\|R\|_{ex} \leq \|R\|_e \leq \|P_\alpha\|_{(L_\alpha^p, A_\alpha^p)} \|R\|_{ex}.$$

The strategy behind the proof of the theorem is to demonstrate the following string of inequalities

$$\mathfrak{b}_S \leq \mathfrak{c}_S \lesssim \overline{\lim}_{m \rightarrow \infty} \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \mathfrak{b}_S \quad (25)$$

$$\|S\|_e \lesssim \overline{\lim}_{m \rightarrow \infty} \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \mathfrak{a}_S \lesssim \|S\|_e. \quad (26)$$

The implied constants in all these estimates depend only on p, α and the dimension of the domain, n and the dimension of the range, d . Combining (25) and (26) we have the theorem. We prove now the first two inequalities in (26).

Fix $f \in A_\alpha^p$ of norm 1 and note that

$$\begin{aligned} \|S_m f\|_{L_\alpha^p}^p &= \sum_{j=m}^{\infty} \left\| M_{1_{F_j}} S T_{\mu 1_{G_j}} f \right\|_{L_\alpha^p}^p \\ &= \sum_{j=m}^{\infty} \left(\frac{\left\| M_{1_{F_j}} S T_{\mu 1_{G_j}} f \right\|_{L_\alpha^p}}{\left\| T_{\mu 1_{G_j}} f \right\|_{A_\alpha^p}} \right)^p \left\| T_{\mu 1_{G_j}} f \right\|_{A_\alpha^p}^p \\ &\leq \sup_{j \geq m} \sup \left\{ \left\| M_{1_{F_j}} S g \right\|_{L_\alpha^p}^p : g \in T_{\mu 1_{G_j}}(A_\alpha^p), \|g\|_{A_\alpha^p} = 1 \right\} \sum_{j \geq m} \left\| T_{\mu 1_{G_j}} f \right\|_{A_\alpha^p}^p \\ &\lesssim \sup_{j \geq m} \sup \left\{ \left\| M_{1_{F_j}} S g \right\|_{L_\alpha^p}^p : g \in T_{\mu 1_{G_j}}(A_\alpha^p), \|g\|_{A_\alpha^p} = 1 \right\}. \end{aligned} \quad (27)$$

In the last step we use the estimate in (24). Since $\text{diam}_\beta G_j \leq \mathfrak{d}$, by selecting $z_j \in G_j$ we have $G_j \subset D(z_j, d)$, and so $T_{\mu 1_{G_j}}(A_\alpha^p) \subset T_{\mu 1_{D(z_j, \mathfrak{d})}}(A_\alpha^p)$. Since z_j approaches the boundary, we can select an additional sequence $0 < \gamma_m < 1$ tending to 1 such that $|z_j| \geq \gamma_m$ when $j \geq m$. Using (27) we find that

$$\|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \sup_{j \geq m} \sup \left\{ \left\| M_{1_{F_j}} S g \right\|_{L_\alpha^p} : g \in T_{\mu 1_{G_j}}(A_\alpha^p), \|g\|_{A_\alpha^p} = 1 \right\} \quad (28)$$

$$\begin{aligned} &\lesssim \sup_{|z_j| \geq \gamma_m} \sup \left\{ \left\| M_{1_{D(z_j, d)}} S g \right\|_{L_\alpha^p} : g \in T_{\mu 1_{D(z_j, \mathfrak{d})}}(A_\alpha^p), \|g\|_{A_\alpha^p} = 1 \right\} \\ &\lesssim \sup_{|z_j| \geq \gamma_m} \sup \left\{ \|S g\|_{L_\alpha^p} : g \in T_{\mu 1_{D(z_j, \mathfrak{d})}}(A_\alpha^p), \|g\|_{A_\alpha^p} = 1 \right\}. \end{aligned} \quad (29)$$

Since $\gamma_m \rightarrow 1$ as $m \rightarrow \infty$, we get

$$\overline{\lim}_{m \rightarrow \infty} \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \mathfrak{a}_S(\mathfrak{d}).$$

From (23) we see that

$$\|S T_\mu\|_{ex} \leq \overline{\lim}_{m \rightarrow \infty} \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} + \epsilon \lesssim \mathfrak{a}_S(\mathfrak{d}) + \epsilon \lesssim \mathfrak{a}_S + \epsilon,$$

giving $\|S T_\mu\|_{ex} \leq \overline{\lim}_{m \rightarrow \infty} \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \mathfrak{a}_S$, since ϵ is arbitrary. Therefore,

$$\|S\|_e \approx \|S T_\mu\|_e \lesssim \|S T_\mu\|_{ex} \leq \overline{\lim}_m \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \mathfrak{a}_S. \quad (30)$$

This gives the first two inequalities in (26). The remaining inequality is simply (22), which was proved in Theorem 5.2.

We now consider (25). If $0 < r < 1$, there exists a positive integer $m(r)$ such that $\bigcup_{j < m(r)} F_j \subset r\mathbb{B}^n$. Then

$$\begin{aligned}
\left\| M_{1_{(r\mathbb{B}^n)^c}} S \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \left\| T_\mu^{-1} \right\|_{\mathcal{L}(A_\alpha^p, A_\alpha^p)}^{-1} &\leq \left\| M_{1_{(r\mathbb{B}^n)^c}} S T_\mu \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \\
&\leq \left\| M_{1_{(r\mathbb{B}^n)^c}} \left(S T_\mu - \sum_{j=1}^{\infty} M_{1_{F_j}} S T_{1_{G_j} \mu} \right) \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \\
&\quad + \left\| M_{1_{(r\mathbb{B}^n)^c}} \sum_{j=1}^{\infty} M_{1_{F_j}} S T_{1_{G_j} \mu} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \\
&\leq \epsilon + \left\| \sum_{j=m(r)}^{\infty} M_{1_{F_j}} S T_{1_{G_j} \mu} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \\
&= \epsilon + \left\| S_{m(r)} \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)}.
\end{aligned}$$

This string of inequalities easily yields

$$\mathbf{c}_S = \overline{\lim}_{r \rightarrow 1} \left\| M_{1_{(r\mathbb{B}^n)^c}} S \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \overline{\lim}_{m \rightarrow \infty} \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)}. \quad (31)$$

Also, (29) gives that

$$\overline{\lim}_{m \rightarrow \infty} \|S_m\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \overline{\lim}_{|z| \rightarrow 1} \left\| M_{1_{D(z,r)}} S \right\|_{\mathcal{L}(A_\alpha^p, L_\alpha^p)} \lesssim \mathbf{b}_S. \quad (32)$$

Combining the trivial inequality $\mathbf{b}_S \leq \mathbf{c}_S$ with (31) and (32) we obtain (25). \square

We can now deduce two results from these theorems. First, we give a way to compute the essential norm of an operator.

Corollary 5.4. *Let $\alpha > -1$ and $1 < p < \infty$ and $S \in \mathcal{T}_{p,\alpha}$. Then*

$$\|S\|_e \approx \sup_{\|f\|_{A_\alpha^p}=1} \overline{\lim}_{|z| \rightarrow 1} \|S_z f\|_{A_\alpha^p}.$$

Proof. It is easy to see from Lemma 4.12 and the compactness of $M_{\mathcal{A}}$ that

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x f\|_{A_\alpha^p} = \overline{\lim}_{|z| \rightarrow 1} \|S_z f\|_{A_\alpha^p}.$$

But then,

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}^n} \|S_x\|_{\mathcal{L}(A_\alpha^p, A_\alpha^p)} = \sup_{\|f\|_{A_\alpha^p}=1} \overline{\lim}_{|z| \rightarrow 1} \|S_z f\|_{A_\alpha^p}.$$

The result then follows from Theorem 5.2. \square

The next is the main result of the paper.

Theorem 5.5. *Let $1 < p < \infty$, $\alpha > -1$ and $S \in \mathcal{L}(A_\alpha^p)$. Then S is compact if and only if $S \in \mathcal{T}_{p,\alpha}$ and $B(S) = 0$ on $\partial\mathbb{B}^n$.*

Proof. If $B(S) = 0$ on $\partial\mathbb{B}^n$, Proposition 4.14 says that $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \mathbb{B}^n$. So, if $S \in \mathcal{T}_{p,\alpha}$, Theorem 5.2 gives that S must be compact.

In the other direction, if S is compact then $B(S) = 0$ on $\partial\mathbb{B}^n$ by (20). So it only remains to show that $S \in \mathcal{T}_{p,\alpha}$. Since every compact operator on A_α^p can be approximated by finite

rank operators, it suffices to show that all rank one operators are in $\mathcal{T}_{p,\alpha}$. Rank one operators have the form $f \otimes g$, given by

$$(f \otimes g)(h) = \langle h, g \rangle_{A_\alpha^2} f,$$

where $f \in A_\alpha^p$, $g \in A_\alpha^q$, and $h \in A_\alpha^p$.

We can further suppose that f and g are polynomials, since the polynomials are dense in A_α^p and A_α^q , respectively (recall that in the vector-valued case a monomial is simply $z^n e$ where e is a constant vector in \mathbb{C}^d). For a vector-valued function f , let \tilde{f} be the matrix-valued function whose diagonal is f and all other entries are zero. That is, the (i, i) entry of \tilde{f} is the i^{th} entry of f and all off diagonal entries are zero. Also, define $\mathbf{1}$ to be the vector in \mathbb{C}^d consisting of all 1's. Consider the following computation:

$$\begin{aligned} \left(T_{\tilde{f}}(\mathbf{1} \otimes \mathbf{1}) T_{\tilde{g}^*} \right) h &= T_{\tilde{f}} \left(\langle P(\tilde{g}^* h), \mathbf{1} \rangle_{A_\alpha^2} \mathbf{1} \right) \\ &= P \left(\langle P(\tilde{g}^* h), \mathbf{1} \rangle_{A_\alpha^2} \tilde{f} \mathbf{1} \right) \\ &= P \left(\langle h, \tilde{g} \mathbf{1} \rangle_{A_\alpha^2} \tilde{f} \mathbf{1} \right) \\ &= \langle h, g \rangle_{A_\alpha^2} f \\ &= (f \otimes g)(h). \end{aligned}$$

So, it suffices to show that $\mathbf{1} \otimes \mathbf{1} \in \mathcal{T}_{p,\alpha}$. Let W be the matrix consisting of all 1's, and let δ_0 be the point mass at 0. Then:

$$\begin{aligned} T_{\delta_0 W} h &= W h(0) \\ &= \left(\sum_{i=1}^d \langle h(0), e_i \rangle_{\mathbb{C}^d} \right) \mathbf{1} \\ &= \int_{\mathbb{B}^n} \langle h(z), \mathbf{1} \rangle_{\mathbb{C}^d} \mathbf{1} dv_\alpha(z) \\ &= \langle h, \mathbf{1} \rangle_{A_\alpha^2} \mathbf{1} \\ &= (\mathbf{1} \otimes \mathbf{1})(h). \end{aligned}$$

By Theorem 4.1, $T_{\delta_0 W}$ is a member of $\mathcal{T}_{p,\alpha}$. □

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